

110.202. Calculus III

2005 Summer

Midterm I

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1. Compute the following limits if they exist:

(a) For $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, we define $f(\mathbf{x}) = \|\mathbf{x}\|$. What is $\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x})$?

(b) $\lim_{(x,y) \rightarrow (0,0)} \sqrt{\left| \frac{2x+y}{x-2y} \right|}$ with $x \neq 2y$.

[Solution]

(a) For $\mathbf{x} \in \mathbb{R}^n$, we can write $\mathbf{x} = (a_1, \dots, a_n)$. Then $f(\mathbf{x}) = \|\mathbf{x}\| = \sqrt{a_1^2 + \dots + a_n^2}$. Since $f(\mathbf{x}) = \|\mathbf{x}\|$ is made by only squares, summations and square root which are all continuous, we know f is continuous. Therefore, $\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0) = \|\mathbf{x}_0\|$.

(b) If we approach $(0, 0)$ along $x = y$, we have

$$\lim_{(y,y) \rightarrow (0,0)} \sqrt{\left| \frac{2y+y}{y-2y} \right|} = \lim_{(y,y) \rightarrow (0,0)} \sqrt{|-3|} = \sqrt{3}.$$

If we approach $(0, 0)$ along $x = 3y$, we have,

$$\lim_{(3y,y) \rightarrow (0,0)} \sqrt{\left| \frac{6y+y}{3y-2y} \right|} = \lim_{(4y,y) \rightarrow (0,0)} \sqrt{|7|} = \sqrt{7}.$$

Therefore, the $\lim_{(x,y) \rightarrow (0,0)} \sqrt{\left| \frac{2x+y}{x-2y} \right|}$ does not exist.

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2. For the function

$$f(x, y, z) = (xy)^z$$

and

$$\mathbf{g}(t) = (2t, t^2, t^3),$$

find ∇f and \mathbf{g}' and evaluate $(f \circ \mathbf{g})'(1)$.

[Solution]

By definition, we have

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (z(xy)^{z-1}y, z(xy)^{z-1}x, xy^z \ln xy).$$

Since $\mathbf{g}(1) = (2, 1, 1)$, we have

$$\begin{aligned} \nabla f|_{(2,1,1)} &= (1 \cdot 2^0 \cdot 1, 1 \cdot 2^0 \cdot 2, (2 \cdot 1)^1 \ln(2 \cdot 1)) \\ &= (1, 2, 2 \ln 2). \end{aligned}$$

By definition,

$$\mathbf{g}'(1) = (2, 2t, 3t^2)|_{t=1} = (2, 2, 3).$$

By chain rule, we have

$$\begin{aligned} (f \circ \mathbf{g})'(1) &= \nabla f(\mathbf{g}(1)) \cdot \mathbf{g}'(1) \\ &= (1, 2, 2 \ln 2) \cdot (2, 2, 3) \\ &= 1 \cdot 2 + 2 \cdot 2 + 2 \ln 2 \cdot 3 \\ &= 6 + 6 \ln 2. \end{aligned}$$

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3. Find the equation for the plane tangent to the surface

$$z = \ln \sqrt{2 + xy}$$

at the point $(1, -1, 0)$.

[Solution]

Let $f(x, y) = \ln \sqrt{2 + xy}$.

The equation of the tangent of the graph of $f(x, y)$ is

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0).$$

We have

$$\begin{aligned} f(1, -1) &= \ln \sqrt{2 + 1(-1)} = 0; \\ \frac{\partial f}{\partial x}(1, -1) &= \frac{1}{\sqrt{2 + xy}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2 + xy}} \cdot y \Big|_{(1,-1)} = 1 \cdot \frac{1}{2} \cdot 1 \cdot (-1) = -\frac{1}{2}; \\ \frac{\partial f}{\partial y}(1, -1) &= \frac{1}{\sqrt{2 + xy}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2 + xy}} \cdot x \Big|_{(1,-1)} = 1 \cdot \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}. \end{aligned}$$

Therefore, the equation is

$$\begin{aligned} z &= 0 + \left(-\frac{1}{2} \right) (x - 1) + \left(\frac{1}{2} \right) (y + 1) \\ &= -\frac{x}{2} + \frac{y}{2} + 1. \end{aligned}$$

So, the equation for the plane tangent to the surface $z = x^y$ at the point $(2, 1, 2)$ is $z = x + 2 \ln 2y - 2 \ln 2$.

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4. Determine the second-order Taylor formula for the function

$$f(x, y) = \sin(xy) + \cos(xy)$$

about the point $(0, 0)$.

[Solution]

We have

$$\frac{\partial f}{\partial x} = \cos(xy) \cdot y - \sin(xy) \cdot y = y \cos(xy) - y \sin(xy)$$

and

$$\frac{\partial f}{\partial y} = \cos(xy) \cdot x - \sin(xy) \cdot x = x \cos(xy) - x \sin(xy).$$

Moreover,

$$\frac{\partial^2 f}{\partial x^2} = y \cdot (-\sin(xy)) \cdot y - y \cdot \cos(xy) \cdot y = -y^2 \sin(xy) - y^2 \cos(xy),$$

$$\frac{\partial^2 f}{\partial y^2} = x \cdot (-\sin(xy)) \cdot x - x \cdot \cos(xy) \cdot x = -x^2 \sin(xy) - x^2 \cos(xy)$$

and

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} \\ &= \cos(xy) + y \cdot (-\sin(xy)) \cdot x - \sin(xy) - y \cos(xy) \cdot x \\ &= \cos(xy) - \sin(xy) - xy \sin(xy) - xy \cos(xy). \end{aligned}$$

Hence, $f(0, 0) = 1$, $\frac{\partial f}{\partial x}\big|_{(0,0)} = 0$, $\frac{\partial f}{\partial y}\big|_{(0,0)} = 0$, $\frac{\partial^2 f}{\partial x^2}\big|_{(0,0)} = 0$, $\frac{\partial^2 f}{\partial y^2}\big|_{(0,0)} = 0$ and $\frac{\partial^2 f}{\partial x \partial y}\big|_{(0,0)} = 1$. Therefore, the second-order

Taylor formula about $(0, 0)$ is

$$\begin{aligned}
 & f(0 + h_1, 0 + h_2) \\
 = & f(0, 0) + \frac{\partial f}{\partial x}(0, 0) h_1 + \frac{\partial f}{\partial y}(0, 0) h_2 \\
 & + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(0, 0) h_1 h_1 + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(0, 0) h_2 h_2 + \frac{\partial^2 f}{\partial x \partial y}(0, 0) h_1 h_2 \\
 & + \mathbf{R}_2(0, 0; h_1, h_2) \\
 = & 1 + (0) h_1 + (0) h_2 + \frac{1}{2} (0) h_1 h_1 + \frac{1}{2} (0) h_2 h_2 + (1) h_1 h_2 \\
 & + \mathbf{R}_2(0, 0; h_1, h_2) \\
 = & 1 + h_1 h_2 + \mathbf{R}_2(0, 0; h_1, h_2) \\
 & \text{where } \frac{\mathbf{R}_2(0, 0; h_1, h_2)}{\|(h_1, h_2)\|} \rightarrow 0 \text{ as } (h_1, h_2) \rightarrow (0, 0).
 \end{aligned}$$

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5. Find the critical points of the function

$$f(x, y) = y \sin(\pi x)$$

and then determine whether they are local maxima, local minima, or saddle points.

[Solution]

We have

$$\nabla f = (\pi y \cos(\pi x), \sin(\pi x)).$$

By setting $\nabla f = 0$, we have $\pi y \cos(\pi x) = 0$ and $\sin(\pi x) = 0$. $\sin(\pi x) = 0$ implies x is an integer. When $\sin(\pi x) = 0$, we know that $\cos(\pi x) = \pm 1$. $\pi y \cos(\pi x) = 0$ implies $y = 0$. Therefore the critical points are $(n, 0)$ where n is an integer.

Moreover, since the discriminant

$$\begin{aligned}
 \Delta(n, 0) &= \left. \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 \right|_{(n, 0)} \\
 &= \left. (-\pi^2 y \sin(\pi x)) \cdot (0) - (\pi \cos(\pi x))^2 \right|_{(n, 0)} \\
 &= 0 - (\pi \cdot (\pm 1))^2 \\
 &= -\pi^2 < 0,
 \end{aligned}$$

we know that $(n, 0)$ are saddle points for all integers n .

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6. Find the absolute maximum and minimum for the function

$$f(x, y) = xy - y + x - 1$$

on the set

$$B = \{(x, y) \mid x^2 + y^2 \leq 2\}.$$

[Solution]

Note that B is closed and boundary. Therefore, absolute maximum and minimum exist.

Write $B = U \cup \partial U$ where $U = \{(x, y) \mid x^2 + y^2 < 2\}$ and $\partial U = \{(x, y) \mid x^2 + y^2 = 2\}$.

Since $\nabla f = (y + 1, x - 1)$, we have a critical point candidate $(1, -1)$. But, $(1)^2 + (-1)^2 = 2$. So, $(1, -1)$ is actually on ∂U . Thus, no critical point on U .

To find the critical points on ∂B , let $f(x, y) = xy - y + x - 1$ and $g(x, y) = x^2 + y^2$. By the method of Lagrange multiplier, we have

$$\nabla f = \lambda \nabla g,$$

that is,

$$(y + 1, x - 1) = \lambda(2x, 2y).$$

So, we have the following system of equations:

$$\begin{cases} y + 1 = 2\lambda x \\ x - 1 = 2\lambda y \\ x^2 + y^2 = 2. \end{cases}$$

First, we consider the case $\lambda = 0$. In this case, $(x, y) = (1, -1)$ which we had found above. If $\lambda \neq 0$, then by adding first two equations together, we have $x + y = 2\lambda(x + y)$. Thus, we have $x + y = 0$ or $\lambda = \frac{1}{2}$. For $x + y = 0$, the third equation gives us $(x, y) = (1, -1)$ or $(-1, 1)$. For $\lambda = \frac{1}{2}$, the first equation becomes $y + 1 = x$. Substitution into the third equation, we have $y = \frac{-1 \pm \sqrt{3}}{2}$. Therefore, we have $(x, y) = \left(\frac{1 + \sqrt{3}}{2}, \frac{-1 + \sqrt{3}}{2}\right)$ or $\left(\frac{1 - \sqrt{3}}{2}, \frac{-1 - \sqrt{3}}{2}\right)$.

Evaluating all four candidates, we have

$$\begin{aligned} f\left(\frac{1 + \sqrt{3}}{2}, \frac{-1 + \sqrt{3}}{2}\right) &= \frac{1}{2}; \\ f\left(\frac{1 - \sqrt{3}}{2}, \frac{-1 - \sqrt{3}}{2}\right) &= \frac{1}{2}; \\ f(1, -1) &= 0; \\ f(-1, 1) &= -4. \end{aligned}$$

Thus, $f\left(\frac{1 + \sqrt{3}}{2}, \frac{-1 + \sqrt{3}}{2}\right) = f\left(\frac{1 - \sqrt{3}}{2}, \frac{-1 - \sqrt{3}}{2}\right) = \frac{1}{2}$ is the absolute maximum and $f(-1, 1) = -4$ is the absolute minimum.

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7. A rectangular box with no top is to have a surface area 16 m^2 . Find the dimensions that maximize its volume.

[Solution]

Let x, y, z be the dimensions of the box with respect to length, wide and height. Let the volume of the box is $V(x, y, z) = xyz$. The surface area of the box is $xy + 2yz + 2xz = 16$. Define $g(x, y, z) = xy + 2yz + 2xz$. By the method of Lagrange multiplier, we have

$$\nabla V = \lambda \nabla g,$$

that is,

$$(yz, xz, xy) = \lambda(y + 2z, x + 2z, 2x + 2y).$$

So, we have the following system of equations:

$$\begin{cases} yz = \lambda(y + 2z) \\ xz = \lambda(x + 2z) \\ xy = 2\lambda(x + y) \\ xy + 2yz + 2xz = 16. \end{cases}$$

First, we claim that $x \neq 0$. If $x = 0$, then $2yz = 16$ and $\lambda(0 + 2z) = 0$. This implies $\lambda = 0$, this means, $yz = 0$ which is a contradiction. Similarly, $y \neq 0$ and $z \neq 0$. Also, we have $x + y \neq 0$, $y + 2z \neq 0$, $x + 2z \neq 0$ and $\lambda \neq 0$. Elimination of λ from the first two equations gives

$$\frac{y}{x} = \frac{y + 2z}{x + 2z}.$$

This implies $x = y$. Similarly, we have $y = 2z$ and $x = 2z$. Hence, $x = y = 2z$. Solving from the last equation, we get $12z^2 = 16$, which means, $z = \sqrt{\frac{16}{12}} = \frac{2}{\sqrt{3}}$. So, $x = y = \frac{4}{\sqrt{3}}$. Since the volume must be bounded (cannot be infinite volume), we know that absolute maximum must exist. Thus, the dimensions $x = y = \frac{4}{\sqrt{3}}$ and $z = \frac{2}{\sqrt{3}}$ gives the maximal volume $\frac{32}{3\sqrt{3}}$.

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8. Is it possible to solve the system of equations

$$\begin{cases} xy^2 + xzu + yv^2 = 3 \\ u^3yz + 2xv - u^2v^2 = 2 \end{cases}$$

for $u(x, y, z)$, $v(x, y, z)$ near $(x, y, z) = (1, 1, 1)$ and $(u, v) = (1, 1)$? If yes, compute $\frac{\partial v}{\partial y}$ at $(x, y, z) = (1, 1, 1)$.

[Solution]

Let

$$F_1 = xy^2 + xzu + yv^2$$

and

$$F_2 = u^3yz + 2xv - u^2v^2.$$

Then the determinant at $(x, y, z) = (1, 1, 1)$ and $(u, v) = (1, 1)$ is

$$\begin{aligned} \Delta &= \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{vmatrix}_{(x,y,z,u,v)=(1,1,1,1,1)} \\ &= \begin{vmatrix} xz & 2yv \\ 3u^2yz - 2uv^2 & 2x - 2u^2v \end{vmatrix}_{(x,y,z,u,v)=(1,1,1,1,1)} \\ &= \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 1 \cdot 0 - 2 \cdot 1 = -2. \end{aligned}$$

Since $\Delta \neq 0$, it is possible to solve for u, v in terms of x, y, z near $(x, y, z) = (1, 1, 1)$ and $(u, v) = (1, 1)$. To compute $\frac{\partial v}{\partial y}$, we first get

$$\frac{\partial F_1}{\partial y} = 2xy + xz \frac{\partial u}{\partial y} + v^2 + 2yv \frac{\partial v}{\partial y} = 0$$

and

$$\frac{\partial F_2}{\partial y} = 3u^2 \frac{\partial u}{\partial y} yz + u^3 z + 2x \frac{\partial v}{\partial y} - 2u \frac{\partial u}{\partial y} v^2 - 2v \frac{\partial v}{\partial y} u^2 = 0.$$

At $(x, y, z) = (1, 1, 1)$ and $(u, v) = (1, 1)$, we have

$$\begin{cases} 3 + \frac{\partial u}{\partial y} + 2 \frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial y} + 1 = 0. \end{cases}$$

Therefore,

$$\begin{cases} \frac{\partial u}{\partial y} = -1 \\ \frac{\partial v}{\partial y} = -1. \end{cases}$$

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9. When $c \neq 0$, the level curve $\{(x, y) \mid f(x, y) = c\}$ of the function

$$f(x, y) = \frac{2x}{x^2 + y^2}$$

is a circle. What is its radius, and where is its center? What happens when $c = 0$?

[Solution]

The level curve is $f(x, y) = c$. So, we have

$$\frac{2x}{x^2 + y^2} = c,$$

that is,

$$cx^2 + cy^2 = 2x.$$

By rewriting it into the standard form of a circle, we have

$$\left(x - \frac{1}{c}\right)^2 + y^2 = \frac{1}{c^2}.$$

Thus, we know the radius is $\frac{1}{|c|}$ and the center is $(\frac{1}{c}, 0)$.

When $c = 0$, the level curve becomes

$$\frac{2x}{x^2 + y^2} = 0,$$

that is,

$$2x = 0$$

or

$$x = 0.$$

So, the level curve is a straight line $x = 0$ when $c = 0$.

