1. Compute the following limits if they exist:
(a) For \( f : \mathbb{R}^n \to \mathbb{R} \) and \( x \in \mathbb{R}^n \), we define \( f(x) = \|x\| \). What is \( \lim_{x \to x_0} f(x) \)?

(b) \( \lim_{(x,y) \to (0,0)} \sqrt{\frac{2x+y}{x-2y}} \) with \( x \neq 2y \).

[Solution]

(a) For \( x \in \mathbb{R}^n \), we can write \( x = (a_1, \ldots, a_n) \). Then \( f(x) = \|x\| = \sqrt{a_1^2 + \cdots + a_n^2} \). Since \( f(x) = \|x\| \) is made by only squares, summations and square root which are all continuous, we know \( f \) is continuous. Therefore, \( \lim_{x \to x_0} f(x) = f(x_0) = \|x_0\| \).

(b) If we approaches \((0, 0)\) along \( x = y \), we have

\[
\lim_{(y,y) \to (0,0)} \sqrt{\frac{2y+y}{y-2y}} = \lim_{(y,y) \to (0,0)} \sqrt{-3} = \sqrt{3}.
\]

If we approaches \((0, 0)\) along \( x = 3y \), we have,

\[
\lim_{(3y,y) \to (0,0)} \sqrt{\frac{6y+y}{3y-2y}} = \lim_{(4y,y) \to (0,0)} \sqrt{7} = \sqrt{7}.
\]

Therefore, the \( \lim_{(x,y) \to (0,0)} \sqrt{\frac{2x+y}{x-2y}} \) does not exist.

2. For the function

\[ f(x, y, z) = (xy)^z \]

and

\[ g(t) = (2t, t^2, t^3), \]

find \( \nabla f \) and \( g' \) and evaluate \( (f \circ g)'(1) \).

[Solution]
By definition, we have
\[ \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (z (xy)^{z-1} y, z (xy)^{z-1} x, xy \ln xy). \]

Since \( g(1) = (2, 1, 1) \), we have
\[
\nabla f \big|_{(2,1,1)} = \left( 1 \cdot 2^0 \cdot 1, 1 \cdot 2^0 \cdot 2, (2 \cdot 1)^1 \ln (2 \cdot 1) \right) = (1, 2, 2 \ln 2).
\]

By definition,
\[ g'(1) = (2, 2t, 3t^2) \big|_{t=1} = (2, 2, 3). \]

By chain rule, we have
\[
(f \circ g)'(1) = \nabla f (g(1)) \cdot g'(1) = (1, 2, 2 \ln 2) \cdot (2, 2, 3) = 1 \cdot 2 + 2 \cdot 2 + 2 \ln 2 \cdot 3 = 6 + 6 \ln 2.
\]

3. Find the equation for the plane tangent to the surface 
\[ z = \ln \sqrt{2 + xy} \]
at the point \((1, -1, 0)\).

[Solution]

Let \( f(x, y) = \ln \sqrt{2 + xy} \).

The equation of the tangent of the graph of \( f(x, y) \) is
\[
z = f(x_0, y_0) + \left[ \frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0).
\]

We have
\[
f(1, -1) = \ln \sqrt{2 + 1(-1)} = 0;
\]
\[
\frac{\partial f}{\partial x}(1, -1) = \frac{1}{\sqrt{2 + xy}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2 + xy}} \cdot y \bigg|_{(1,-1)} = 1 \cdot \frac{1}{2} \cdot 1 \cdot (-1) = -\frac{1}{2};
\]
\[
\frac{\partial f}{\partial y}(1, -1) = \frac{1}{\sqrt{2 + xy}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2 + xy}} \cdot x \bigg|_{(1,-1)} = 1 \cdot \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}.
\]

Therefore, the equation is
\[
z = 0 + \left( -\frac{1}{2} \right)(x - 1) + \left( \frac{1}{2} \right)(y + 1) = -\frac{x}{2} + \frac{y}{2} + 1.
\]
So, the equation for the plane tangent to the surface \( z = x^y \) at the point \((2, 1, 2)\) is \( z = x + 2 \ln 2y - 2 \ln 2 \).

4. Determine the second-order Taylor formula for the function

\[ f (x, y) = \sin (xy) + \cos (xy) \]

about the point \((0, 0)\).

**[Solution]**

We have

\[ \frac{\partial f}{\partial x} = \cos (xy) \cdot y - \sin (xy) \cdot y = y \cos (xy) - y \sin (xy) \]

and

\[ \frac{\partial f}{\partial y} = \cos (xy) \cdot x - \sin (xy) \cdot x = x \cos (xy) - x \sin (xy) \]

Moreover,

\[ \frac{\partial^2 f}{\partial x^2} = y \cdot (- \sin (xy)) \cdot y - y \cdot \cos (xy) \cdot y = -y^2 \sin (xy) - y^2 \cos (xy) \]

\[ \frac{\partial^2 f}{\partial y^2} = x \cdot (- \sin (xy)) \cdot x - x \cdot \cos (xy) \cdot x = -x^2 \sin (xy) - x^2 \cos (xy) \]

and

\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \cos (xy) + y \cdot (- \sin (xy)) \cdot x - \sin (xy) - y \cdot \cos (xy) \cdot x \]

\[ = \cos (xy) - \sin (xy) - xy \sin (xy) - xy \cos (xy) \]

Hence, \( f (0, 0) = 1 \), \( \frac{\partial f}{\partial x} \bigg|_{(0,0)} = 0 \), \( \frac{\partial f}{\partial y} \bigg|_{(0,0)} = 0 \), \( \frac{\partial^2 f}{\partial x^2} \bigg|_{(0,0)} = 0 \), \( \frac{\partial^2 f}{\partial y^2} \bigg|_{(0,0)} = 0 \) and \( \frac{\partial^2 f}{\partial x \partial y} \bigg|_{(0,0)} = 1 \). Therefore, the second-order
Taylor formula about \((0, 0)\) is
\[
f (0 + h_1, 0 + h_2) = f (0, 0) + \frac{\partial f}{\partial x} (0, 0) h_1 + \frac{\partial f}{\partial y} (0, 0) h_2 \\
+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (0, 0) h_1 h_1 + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} (0, 0) h_2 h_2 + \frac{\partial^2 f}{\partial x \partial y} (0, 0) h_1 h_2 \\
+ R_2 (0, 0; h_1, h_2)
\]
\[
= 1 + (0) h_1 + (0) h_2 + \frac{1}{2} (0) h_1 h_1 + \frac{1}{2} (0) h_2 h_2 + (1) h_1 h_2 \\
+ R_2 (0, 0; h_1, h_2)
\]
where \(R_2 (0, 0; h_1, h_2) \rightarrow 0\) as \((h_1, h_2) \rightarrow (0, 0)\).

5. Find the critical points of the function
\[
f (x, y) = y \sin (\pi x)
\]
and then determine whether they are local maxima, local minima, or saddle points.

[Solution]
We have
\[
\nabla f = (\pi y \cos (\pi x), \sin (\pi x)).
\]
By setting \(\nabla f = 0\), we have \(\pi y \cos (\pi x) = 0\) and \(\sin (\pi x) = 0\).
\(\sin (\pi x) = 0\) implies \(x\) is an integer. When \(\sin (\pi x) = 0\), we know that \(\cos (\pi x) = \pm 1\). \(\pi y \cos (\pi x) = 0\) implies \(y = 0\).
Therefore the critical points are \((n, 0)\) where \(n\) is an integer.
Moreover, since the discriminant
\[
\Delta (n, 0) = \left. \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 \right|_{(n, 0)}
\]
\[
= (-\pi^2 y \sin (\pi x)) \cdot (0) - (\pi \cos (\pi x))^2 \mid_{(n, 0)}
\]
\[
= 0 - (\pi \cdot (\pm 1))^2
\]
\[
= -\pi^2 < 0,
\]
we know that \((n, 0)\) are saddle points for all integers \(n\).

6. Find the absolute maximum and minimum for the function
\[
f (x, y) = xy - y + x - 1
\]
on the set

\[ B = \{(x, y) \mid x^2 + y^2 \leq 2\}. \]

**[Solution]**

Note that \( B \) is closed and boundary. Therefore, absolute maximum and minimum exist.

Write \( B = U \cup \partial U \) where \( U = \{(x, y) \mid x^2 + y^2 < 2\} \) and \( \partial U = \{(x, y) \mid x^2 + y^2 = 2\} \).

Since \( \nabla f = (y + 1, x - 1) \), we have a critical point candidate \((1, -1)\). But, \((1)^2 + (-1)^2 = 2\). So, \((1, -1)\) is actually on \( \partial U \). Thus, no critical point on \( U \).

To find the critical points on \( \partial B \), let \( f(x, y) = xy - y + x - 1 \) and \( g(x, y) = x^2 + y^2 \). By the method of Lagrange multiplier, we have

\[ \nabla f = \lambda \nabla g, \]

that is,

\[(y + 1, x - 1) = \lambda (2x, 2y).\]

So, we have the following system of equations:

\[
\begin{align*}
  y + 1 &= 2\lambda x \\
  x - 1 &= 2\lambda y \\
  x^2 + y^2 &= 2.
\end{align*}
\]

First, we consider the case \( \lambda = 0 \). In this case, \((x, y) = (1, -1)\) which we had found above. If \( \lambda \neq 0 \), then by adding first two equations together, we have \( x + y = 2\lambda (x + y) \). Thus, we have \( x + y = 0 \) or \( \lambda = \frac{1}{2} \). For \( x + y = 0 \), the third equation gives us \((x, y) = (1, -1)\) or \((-1, 1)\). For \( \lambda = \frac{1}{2} \), the first equation becomes \( y + 1 = x \). Substitution into the third equation, we have \( y = -1+\sqrt{3} \). Therefore, we have \((x, y) = \left(\frac{1+\sqrt{3}}{2}, -1+\sqrt{3}\right)\) or \( \left(\frac{1-\sqrt{3}}{2}, -1-\sqrt{3}\right)\).

Evaluating all four candidates, we have

\[
\begin{align*}
  f \left(\frac{1+\sqrt{3}}{2}, -1+\sqrt{3}\right) &= \frac{1}{2}; \\
  f \left(\frac{1-\sqrt{3}}{2}, -1-\sqrt{3}\right) &= \frac{1}{2}; \\
  f (1, -1) &= 0; \\
  f (-1, 1) &= -4.
\end{align*}
\]

Thus, \( f \left(\frac{1+\sqrt{3}}{2}, -1+\sqrt{3}\right) = f \left(\frac{1-\sqrt{3}}{2}, -1-\sqrt{3}\right) = \frac{1}{2} \) is the absolute maximum and \( f (-1, 1) = -4 \) is the absolute minimum.
7. A rectangular box with no top is to have a surface area $16 \text{ m}^2$. Find the dimensions that maximize its volume.

**[Solution]**

Let $x, y, z$ be the dimensions of the box with respect to length, wide and height. Let the volume of the box is $V(x, y, z) = xyz$. The surface area of the box is $xy + 2yz + 2xz = 16$. Define $g(x, y, z) = xy + 2yz + 2xz$. By the method of Lagrange multiplier, we have

\[

\nabla V = \lambda \nabla g,
\]

that is,

\[
(yz, xz, xy) = \lambda (y + 2z, x + 2z, 2x + 2y).
\]

So, we have the following system of equations:

\[
\begin{align*}
    yz &= \lambda (y + 2z) \\
    xz &= \lambda (x + 2z) \\
    xy &= 2\lambda (x + y) \\
    xy + 2yz + 2xz &= 16.
\end{align*}
\]

First, we claim that $x \neq 0$. If $x = 0$, then $2yz = 16$ and $\lambda (0 + 2z) = 0$. This implies $\lambda = 0$, this means, $yz = 0$ which is a contradiction. Similarly, $y \neq 0$ and $z \neq 0$. Also, we have $x + y \neq 0, y + 2z \neq 0, x + 2z \neq 0$ and $\lambda \neq 0$. Elimination of $\lambda$ from the first two equations gives

\[
\frac{y}{x} = \frac{y + 2z}{x + 2z}.
\]

This implies $x = y$. Similarly, we have $y = 2z$ and $x = 2z$. Hence, $x = y = 2z$. Solving from the last equation, we get $12z^2 = 16$, which means, $z = \sqrt{\frac{16}{12}} = \frac{2}{\sqrt{3}}$. So, $x = y = \frac{4}{\sqrt{3}}$. Since the volume must be bounded (cannot be infinite volume), we know that absolute maximum must exist. Thus, the dimensions $x = y = \frac{4}{\sqrt{3}}$ and $z = \frac{2}{\sqrt{3}}$ gives the maximal volume $\frac{32}{3\sqrt{3}}$.

8. Is it possible to solve the system of equations

\[
\begin{align*}
    xy^2 + xzu + yv^2 &= 3 \\
    u^3yz + 2xv - u^2v^2 &= 2
\end{align*}
\]

for $u(x, y, z), v(x, y, z)$ near $(x, y, z) = (1, 1, 1)$ and $(u, v) = (1, 1)$? If yes, compute $\frac{\partial u}{\partial y}$ at $(x, y, z) = (1, 1, 1)$.

**[Solution]**
Let
\[ F_1 = xy^2 + xzu + yv^2 \]
and
\[ F_2 = u^3yz + 2xv - u^2v^2. \]
Then the determinant at \((x, y, z) = (1, 1, 1)\) and \((u, v) = (1, 1)\) is
\[
\Delta = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix}_{(x, y, z, u, v) = (1, 1, 1, 1, 1)}
\]
\[
= \begin{vmatrix} xz & 2yv \\ 3u^2yz - 2uv^2 & 2x - 2u^2v \end{vmatrix}_{(x, y, z, u, v) = (1, 1, 1, 1, 1)}
\]
\[
= \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 1 \cdot 0 - 2 \cdot 1 = -2.
\]
Since \(\Delta \neq 0\), it is possible to solve for \(u, v\) in terms of \(x, y, z\) near \((x, y, z) = (1, 1, 1)\) and \((u, v) = (1, 1)\). To compute \(\frac{\partial u}{\partial y}\), we first get
\[
\frac{\partial F_1}{\partial y} = 2xy + xz \frac{\partial u}{\partial y} + v^2 + 2yv \frac{\partial v}{\partial y} = 0
\]
and
\[
\frac{\partial F_2}{\partial y} = 3u^2 \frac{\partial u}{\partial y} yz + u^3z + 2x \frac{\partial v}{\partial y} - 2u \frac{\partial u}{\partial y} v^2 - 2v \frac{\partial v}{\partial y} u^2 = 0.
\]
At \((x, y, z) = (1, 1, 1)\) and \((u, v) = (1, 1)\), we have
\[
\begin{cases}
3 + \frac{\partial u}{\partial y} + 2 \frac{\partial v}{\partial y} = 0 \\
\frac{\partial u}{\partial y} + 1 = 0.
\end{cases}
\]
Therefore,
\[
\begin{cases}
\frac{\partial u}{\partial y} = -1 \\
\frac{\partial v}{\partial y} = -1.
\end{cases}
\]

9. When \(c \neq 0\), the level curve \(\{(x, y) \mid f(x, y) = c\}\) of the function
\[ f(x, y) = \frac{2x}{x^2 + y^2} \]
is a circle. What it its radius, and where is its center? What happens when \(c = 0\)?

[Solution]
The level curve is \(f(x, y) = c\). So, we have
\[
\frac{2x}{x^2 + y^2} = c,
\]
that is,

\[ cx^2 + cy^2 = 2x. \]

By rewriting it into the standard form of a circle, we have

\[ \left( x - \frac{1}{c} \right)^2 + y^2 = \frac{1}{c^2}. \]

Thus, we know the radius is \( \frac{1}{|c|} \) and the center is \( \left( \frac{1}{c}, 0 \right) \).

When \( c = 0 \), the level curve becomes

\[ \frac{2x}{x^2 + y^2} = 0, \]

that is,

\[ 2x = 0 \]

or

\[ x = 0. \]

So, the level curve is a straight line \( x = 0 \) when \( c = 0 \).