MA261-A Calculus III

2006 Fall

Homework 4 Solutions

Due 9/29/2006 8:00AM

9.7 #14 Describe in words the surface $\theta = \frac{\pi}{3}$.

[Solution]

A half-plane in the positive x and y territory (See Figure 8 in Page 687).

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9.7 #18 Identify the surface $\rho = 2\cos\phi$.

[Solution]

We see ρ and ϕ. It suggests that we are using spherical coordinates.

First notice that since ρ must be bigger than or equal to 0, $2\cos\phi \geq 0$. This implies that $0 \leq \phi \leq \frac{\pi}{2}$.

Since

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases},$$

and $\rho = 2\cos\phi$, we have

$$\begin{cases} x = 2\cos\phi\sin\phi\cos\theta = \sin 2\phi\cos\theta \\ y = 2\cos\phi\sin\phi\sin\theta = \sin 2\phi\sin\theta \\ z = 2\cos\phi\cos\phi = 2\cos^2\phi \end{cases},$$

where $0 \leq \phi \leq \frac{\pi}{2}$ and $0 \leq \theta \leq 2\pi$.

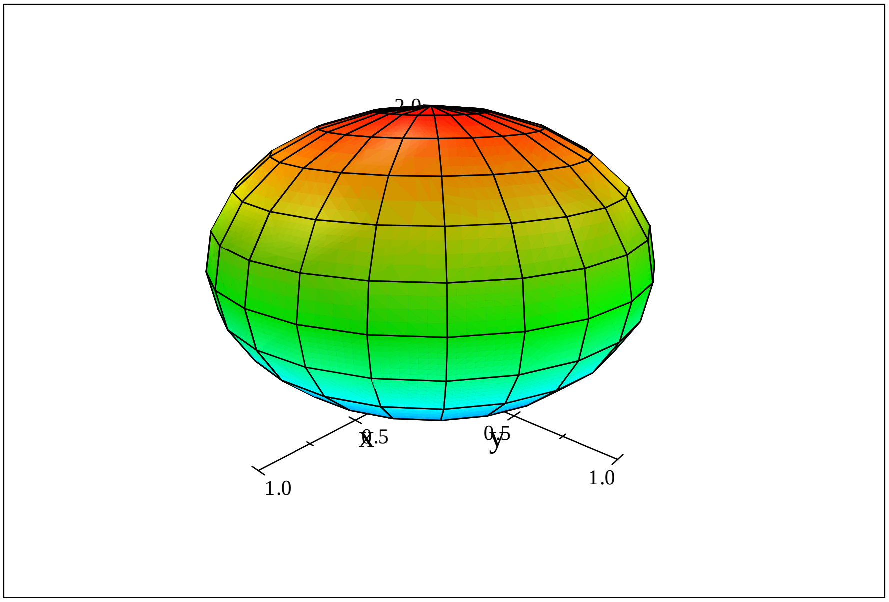
Since $z = 2\cos^2\phi$, we know that $0 \leq z \leq 2$ and when $\phi = 0$, $z = 2$ and when $\phi = \frac{\pi}{2}$, $z = 0$. If we treat ϕ as a scanning line, it scans from $z = 2$ to $z = 0$.

By fixing an angle ϕ, since

$$\begin{cases} x = 2\cos\phi\sin\phi\cos\theta = \sin 2\phi\cos\theta \\ y = 2\cos\phi\sin\phi\sin\theta = \sin 2\phi\sin\theta \end{cases},$$

we know that $\sin 2\phi$ plays the role of the radius of a circle at the height $z = 2\cos^2\phi$. When $\phi = 0$, $\sin 2\phi = 0$, when $\phi = \frac{\pi}{4}$, $\sin 2\phi = 1$, and when $\phi = \frac{\pi}{2}$, $\sin 2\phi = 0$. Thus, the radius of the circles starts from 0 and then gets bigger until the half-way to 1, and then comes back to 0. It makes our graph looked like a sphere.

Here is the graph:



■

9.7 #20 Identify the surface $r^2 - 2z^2 = 4$.

[Solution]

We see r and z. It suggests that we are using cylindrical coordinates.

In cylindrical coordinates, we have $r^2 = x^2 + y^2$. Thus, our equation becomes $x^2 + y^2 - 2z^2 = 4$. By dividing by 4, we get

$$\frac{x^2}{4} + \frac{y^2}{4} - \frac{z^2}{2} = 1.$$

By looking at Table 2 in page 682, we identify this surface as a hyperboloid of one sheet.

■

9.7 #28 Sketch the solid described by $2 \le \rho \le 3$, $\frac{\pi}{2} \le \phi \le \pi$.

[Solution]

We see ρ and ϕ. It suggests that we are using spherical coordinates.

First, let us see when $\rho = 2$. $\frac{\pi}{2} \le \phi \le \pi$ tells us that we have only bottom half of the sphere with radius 2 (looks like a bowl.)

Similarly, when $\rho = 3$. $\frac{\pi}{2} \le \phi \le \pi$ tells us that we have only bottom half of the sphere with radius 3 (looks like a bigger bowl.)

Thus, the solid is everything in between. It looks like a hard-cooked egg without york.

■

9.7 #32 (a) Find inequalities that describe a hollow ball with diameter 30 cm and thickness 0.5 cm. Explain how you have positioned the coordinate system that you have chosen.

(b) Suppose the ball is cut in half. Write inequalities that describe one of the halves.

[Solution]

(a) If we position the center of the ball in the origin of a x-y-z coordinate system. A solid ball with diameter 30 cm can be described as $x^2 + y^2 + z^2 \le 15^2$.

To get the thickness 0.5 cm, we need to take off a solid ball with diameter $30 - 0.5 \times 2 = 29$ cm which can be described as $x^2 + y^2 + z^2 \le 14.5^2$.

So, a hollow ball with diameter 30 cm and thickness 0.5 cm can be described as

$$14.5^2 \le x^2 + y^2 + z^2 \le 15^2.$$

(b) If we cut this ball through the xy plane, we know that half ball has either $z \geq 0$ or $z \leq 0$. So, the upper half can be described as

$$14.5^2 \leq x^2 + y^2 + z^2 \leq 15^2 \text{ and } z \geq 0.$$

Also, the lower half can be described as

$$14.5^2 \leq x^2 + y^2 + z^2 \leq 15^2 \text{ and } z \leq 0.$$

■

9.7 #36 The latitude and longitude of a point P in the Northern Hemisphere are related to spherical coordinates ρ, θ, ϕ as follows. We take the origin to be the center of the Earth and the positive z-axis to pass through the North Pole. The positive x-axis passes through the point where the prime meridian (the meridian through Greenwich, England) intersects the equator. Then the latitude of P is $\alpha = 90° - \phi°$ and the longitude is $\beta = 360° - \theta°$. Find the great circle distance from Los Angeles (lat. 34.06°N, long. 118.25°W) to Montréal (lat. 45.50°N, long. 73.60°W). Take the radius of the earth to be 3960 mi. (A *great circle* is the circle of intersection of a sphere and a plane through the center of the sphere.)

[Solution]

The ρ, θ, ϕ-coordinate of Los Angeles is

$$(\rho, \theta, \phi) = (3960, 360 - 118.25, 90 - 34.06) = (3960, 241.75, 55.94).$$

Thus, the x, y, z-coordinate of Los Angeles is

$$\begin{aligned} & (x, y, z) \\ = \; & (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \\ = \; & (3960 \sin (55.94) \cos (241.75), 3960 \sin (55.94) \sin (241.75), 3960 \cos (55.94)) \\ = \; & (2237.9, -344.26, 3248.8). \end{aligned}$$

Similarly, the ρ, θ, ϕ-coordinate of Montréal is

$$(\rho, \theta, \phi) = (3960, 360 - 73.60, 90 - 45.50) = (3960, 286.4, 44.5).$$

Thus, the x, y, z-coordinate of Montréal is

$$\begin{aligned} & (x, y, z) \\ = \; & (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \\ = \; & (3960 \sin (44.5) \cos (286.4), 3960 \sin (44.5) \sin (286.4), 3960 \cos (44.5)) \\ = \; & (-1705.5, -965.36, 3441.1). \end{aligned}$$

Let $\vec{v}$ be the vector from the origin (center of the earth) to Los Angeles and $\vec{w}$ be the vector from the origin (center of the earth) to Montréal. By using the dot product, we have $\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta$, where θ is the angle between these two vectors which lie on a great circle. So,

$$\begin{aligned} \theta &= \cos^{-1} \frac{(2237.9, -344.26, 3248.8) \cdot (-1705.5, -965.36, 3441.1)}{|(2237.9, -344.26, 3248.8)| \, |(-1705.5, -965.36, 3441.1)|} \\ &= \cos^{-1} \frac{7695000}{3960 \times 2960} \\ &= 0.85465. \end{aligned}$$

Since $\frac{\theta}{360} = \frac{0.85465}{2\pi}$, we know that θ is also $48.968°$. Therefore, the great circle distance D is

$$\frac{48.968}{360} = \frac{D}{2\pi\,(3960)},$$

that is, $D = \frac{48.968}{360} \times 2\pi\,(3960) = 3384.4$.

■

10.1 #2 Find the domain of the vector function

$$\mathbf{r}\,(t) = \frac{t-2}{t+2}\mathbf{i} + \sin t\mathbf{j} + \ln\left(9 - t^2\right)\mathbf{k}.$$

[Solution]

For the x-component function $\frac{t-2}{t+2}$, $t \neq -2$. For the y-component function $\sin t$, there is no restriction of t. For the z-component function $\ln\left(9 - t^2\right)$, we need to have $9 - t^2 > 0$, or $(3-t)(3+t) > 0$. This implies that $t > 3$ or $t < -3$. Thus, the domain is

$$t > 3 \text{ or } t < -3.$$

■

10.1 #4 Find the limit

$$\lim_{t\to\infty}\left\langle \arctan t, e^{-2t}, \frac{\ln t}{t}\right\rangle.$$

[Solution]

$$\lim_{t\to\infty}\left\langle \arctan t, e^{-2t}, \frac{\ln t}{t}\right\rangle = \left\langle \lim_{t\to\infty}\arctan t, \lim_{t\to\infty} e^{-2t}, \lim_{t\to\infty}\frac{\ln t}{t}\right\rangle = \left\langle \frac{\pi}{2}, 0, 0\right\rangle.$$

■

10.1 #10 Sketch the curve with the given vector equation

$$\mathbf{r}\,(t) = t^2\mathbf{i} + t\mathbf{j} + 2\mathbf{k}.$$

Indicate with an arrow the direction in which t increases.

[Solution]

Notice that $z = 2$. In xy-plane, we have $x = t^2$ and $y = t$. This implies that $x = y^2$. So, the curve is a parabolic curve which concave up to the positive y-axis at the height $z = 2$.

When t increase, it goes to the positive y-axis direction.

■

10.1 #12 Sketch the curve with the given vector equation

$$\mathbf{r}\,(t) = \cos t\mathbf{i} - \cos t\mathbf{j} + \sin t\mathbf{k}.$$

Indicate with an arrow the direction in which t increases.

[Solution]

Notice that we have $x = \cos t$ and $z = \sin t$. This implies that we have a circle when projects into the xz-plane. So, it looks like a helix type of curve. Also, since $-1 \leq \cos t \leq 1$, this helix-like curve is within $-1 \leq y \leq 1$.

Consider a cylinder $x^2 + z^2 = 1$ with $-1 \leq y \leq 1$. When $t = 0$, $\mathbf{r}\,(0) = 1\mathbf{i} - 1\mathbf{j} + 0\mathbf{k}$, it is the bottom. As t increases, $-\cos t$ increases. So, the curve travels up through the helix-like curve. After it reaches the top as $t = \pi$, the curve becomes travelling down through the helix-like curve. And, repeat when reaches the bottom ($t = 2\pi$).

■

10.1 #14 Find a vector equation and parametric equations for the line segment that joins $P(1,0,1)$ to $Q(2,3,1)$.

[Solution]

The directional vector is $\overrightarrow{PQ} = \langle 2,3,1\rangle - \langle 1,0,1\rangle = \langle 1,3,0\rangle$. Thus, the vector equation is

$$\langle 1,0,1\rangle + t\langle 1,3,0\rangle, \text{ where } 0 \le t \le 1.$$

Also, the parametric equations are

$$\begin{cases} x = 1+t \\ y = 3t \\ z = 1 \end{cases}, \text{ where } 0 \le t \le 1.$$

■

10.1 #24 Show that the curve with parametric equations $x = \sin t$, $y = \cos t$, $z = \sin^2 t$ is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$. Use this fact to help sketch the curve.

[Solution]

The surface $x^2 + y^2 = 1$ can be described as $(\sin t, \cos t, z)$. The intersection of two surface will have points of this form and satisfy $z = x^2$. Thus, the points in the intersection looks like $(\sin t, \cos t, \sin^2 t)$. Also, when pluging $x = \sin t$, $y = \cos t$, $z = \sin^2 t$ into both surfaces, they are both satisfied. So, we can conclude that the curve with parametric equations $x = \sin t$, $y = \cos t$, $z = \sin^2 t$ is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$.

$x^2 + y^2 = 1$ is a cylinder. $z = x^2$ is a parabolic curve in the xz plane. So, our curve looks like a parabolic curve on the surface of the cylinder $x^2 + y^2 = 1$.

■

10.1 #32 Find a vector function that represents the curve of intersection of the cylinder $x^2+y^2 = 4$ and the surface $z = xy$.

[Solution]

The cylinder $x^2 + y^2 = 4$ can be described as $(2\cos\theta, 2\sin\theta, z)$, where $0 \le \theta \le 2\pi$. The intersection with $z = xy$ makes $z = 4\sin\theta\cos\theta$. Thus, we can write down the vector function

$$\mathbf{r}(t) = \langle 2\cos\theta, 2\sin\theta, 4\sin\theta\cos\theta\rangle.$$

■

10.1 #38 Two particles travel along the space curves

$$\mathbf{r}_1(t) = \langle t, t^2, t^3\rangle \text{ and } \mathbf{r}_2(t) = \langle 1+2t, 1+6t, 1+14t\rangle.$$

Do the particles collide? Do their paths intersect?

[Solution]

Let us re-write $\mathbf{r}_2$ as $\mathbf{r}_2(s) = \langle 1+2s, 1+6s, 1+14s\rangle$. If these two particles collide, we will have at least a pair of (t,s) such that $\mathbf{r}_1(t) = \mathbf{r}_2(s)$. This implies that

$$\begin{cases} t = 1+2s \\ t^2 = 1+6s \\ t^3 = 1+14s \end{cases}.$$

By putting the first equation into the second one, we have $(1+2s)^2 = 1+6s$. This tells us that $s = 0$ or $\frac{1}{2}$.

When $s = 0$, $t = 1$. This pair satisfies the third equation. So, it represents a collision point.

When $s = \frac{1}{2}$, $t = 2$. This pair also satisfies the third equation. So, it represents another collision point.

Thus, these two particles collide at $(1, 1, 1)$ and $(2, 4, 8)$.

■

10.2 #4 $\mathbf{r}(t) = \langle 1 + t, \sqrt{t} \rangle$

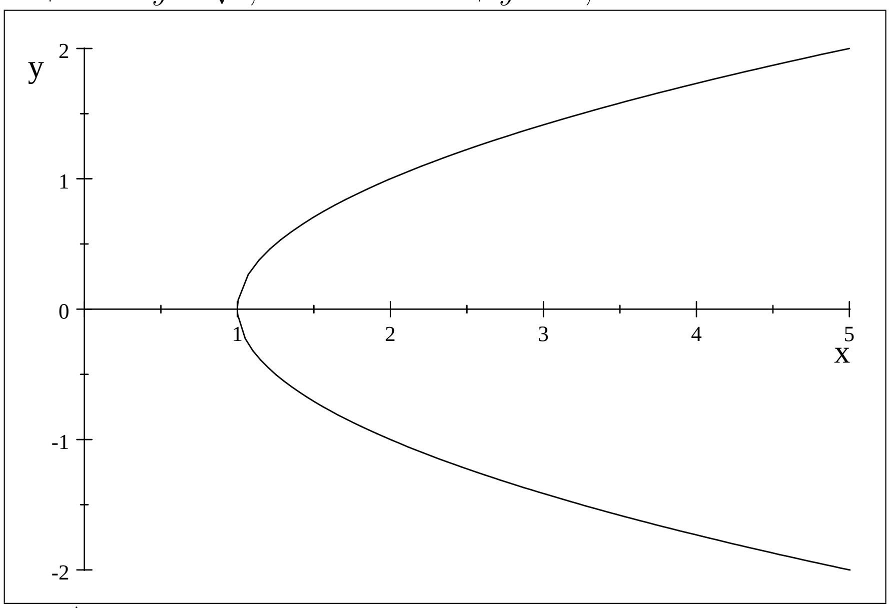
(a) Sketch the plane curve.

(b) Find $\mathbf{r}'(t)$.

(c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for $t = 1$.

[Solution]

(a) Since $x = 1 + t$ and $y = \sqrt{t}$, we have $x = 1 + y^2$. So, the curve is



(b) $\mathbf{r}'(t) = \left\langle 1, \frac{1}{2\sqrt{t}} \right\rangle$.

(c) When $t = 1$, $\mathbf{r}(1) = \langle 2, 1 \rangle$ and $\mathbf{r}'(1) = \langle 1, \frac{1}{2} \rangle$.

■

10.2 #10 Find the derivative of the vector function

$$\mathbf{r}(t) = \langle \cos 3t, t, \sin 3t \rangle .$$

[Solution]

$$\mathbf{r}'(t) = \langle -3 \sin 3t, 1, 3 \cos 3t \rangle .$$

■

10.2 #14 Find the derivative of the vector function

$$\mathbf{r}(t) = t\mathbf{a} \times (\mathbf{b} + t\mathbf{c}) .$$

[Solution]

$$\begin{aligned}\mathbf{r}'(t) &= (t\mathbf{a})' \times (\mathbf{b} + t\mathbf{c}) + t\mathbf{a} \times (\mathbf{b} + t\mathbf{c})' \\ &= \mathbf{a} \times (\mathbf{b} + t\mathbf{c}) + t\mathbf{a} \times \mathbf{c} \\ &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times t\mathbf{c} + t\mathbf{a} \times \mathbf{c} \\ &= \mathbf{a} \times \mathbf{b} + 2t(\mathbf{a} \times \mathbf{c}).\end{aligned}$$

■

10.2 #16 Find the unit tangent vector $\mathbf{T}(t)$ of

$$\mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \tan t\mathbf{k}$$

at the point $t = \frac{\pi}{4}$.

[Solution]

We have

$$\mathbf{r}'(t) = 2\cos t\mathbf{i} - 2\sin t\mathbf{j} + \sec^2 t\mathbf{k}.$$

Thus,

$$\begin{aligned}|\mathbf{r}'(t)| &= \sqrt{(2\cos t)^2 + (-2\sin t)^2 + (\sec^2 t)^2} \\ &= \sqrt{4\cos^2 t + 4\sin^2 t + \sec^4 t} \\ &= \sqrt{4 + \sec^4 t}.\end{aligned}$$

So, the tangent vector is

$$\begin{aligned}\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{4+\sec^4 t}}\left(2\cos t\mathbf{i} - 2\sin t\mathbf{j} + \sec^2 t\mathbf{k}\right) \\ &= \frac{2\cos t}{\sqrt{4+\sec^4 t}}\mathbf{i} - \frac{2\sin t}{\sqrt{4+\sec^4 t}}\mathbf{j} + \frac{\sec^2 t}{\sqrt{4+\sec^4 t}}\mathbf{k}.\end{aligned}$$

■

10.2 #20 Find parametric equations for the tangent line to the curve $x = 2\cos t$, $y = 2\sin t$, and $z = 4\cos 2t$ at the point $\left(\sqrt{3}, 1, 2\right)$. Illustrate by graphing both the curve and the tangent line on a common screen.

[Solution]

Let the curve be represented by a vector function $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\cos 2t\rangle$.

When $t = \frac{\pi}{6}$, we have the point $\left(\sqrt{3}, 1, 2\right)$. So, the directional vector of the tangent line at $\left(\sqrt{3}, 1, 2\right)$ is

$$\mathbf{r}'\left(\frac{\pi}{6}\right) = \langle -2\sin t, 2\cos t, -8\sin 2t\rangle|_{t=\frac{\pi}{6}} = \left\langle -1, \sqrt{3}, -4\sqrt{3}\right\rangle.$$

Thus, the parametric equations for the tangent line is

$$\begin{cases} x = \sqrt{3} + (-1)t = \sqrt{3} - t \\ y = 1 + \left(\sqrt{3}\right)t = 1 + \sqrt{3}t \\ z = 2 + \left(-4\sqrt{3}\right)t = 2 - 4\sqrt{3}t \end{cases}.$$

■

10.2 #28 At what point do the curves $\mathbf{r}_1(t) = \langle t, 1-t, 3+t^2\rangle$ and $\mathbf{r}_2(s) = \langle 3-s, s-2, s^2\rangle$ intersect? Find their angle of intersection correct to the nearest angle.

[Solution]

To get the intersection points, we assume that $\mathbf{r}_1(t) = \mathbf{r}_2(s)$. This gives us a pair $(t, s) = (1, 2)$. So, at the point $(1, 0, 4)$, they intersect.

The tangent vector of $\mathbf{r}_1(t)$ at $(1, 0, 4)$ is $\mathbf{r}_1'(1) = \langle 1, -1, 2t\rangle|_{t=1} = \langle 1, -1, 2\rangle$. The tangent vector of $\mathbf{r}_2(t)$ at $(1, 0, 4)$ is $\mathbf{r}_2'(2) = \langle -1, 1, 2s\rangle|_{s=2} = \langle -1, 1, 4\rangle$. Thus, the angle θ satisfies $\mathbf{r}_1'(1) \cdot \mathbf{r}_2'(2) = |\mathbf{r}_1'(1)| |\mathbf{r}_2'(2)| \cos\theta$. It tells us that

$$\cos\theta = \frac{\langle 1, -1, 2\rangle \cdot \langle -1, 1, 4\rangle}{\left(\sqrt{1^2 + (-1)^2 + 2^2}\right)\left(\sqrt{(-1)^2 + 1^2 + 4^2}\right)} = \frac{1}{\sqrt{3}}.$$

So, the angle is $\cos^{-1}\frac{1}{\sqrt{3}} = 0.95532 \approx 55°$.

■

10.2 #32 Evalute the integral

$$\int_1^2 \left(t^2\mathbf{i} + t\sqrt{t-1}\mathbf{j} + t\sin\pi t\mathbf{k}\right) dt$$

[Solution]

$$\begin{aligned}
&\int_1^2 \left(t^2\mathbf{i} + t\sqrt{t-1}\mathbf{j} + t\sin\pi t\mathbf{k}\right) dt \\
=\ &\left(\int_1^2 t^2 dt\right)\mathbf{i} + \left(\int_1^2 t\sqrt{t-1}dt\right)\mathbf{j} + \left(\int_1^2 t\sin\pi t dt\right)\mathbf{k} \\
=\ &\frac{7}{3}\mathbf{i} + \frac{16}{15}\mathbf{j} - \frac{3}{\pi}\mathbf{k}.
\end{aligned}$$

■

10.2 #36 Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = t\mathbf{i} + e^t\mathbf{j} + te^t\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

[Solution]

Since $\mathbf{r}(t)$ is the antiderivative of $\mathbf{r}'(t)$, we have

$$\begin{aligned}
\mathbf{r}(t) &= \int \mathbf{r}'(t)\,dt = \int \left(t\mathbf{i} + e^t\mathbf{j} + te^t\mathbf{k}\right) dt = \left(\int tdt\right)\mathbf{i} + \left(\int e^t dt\right)\mathbf{j} + \left(\int te^t dt\right)\mathbf{k} \\
&= \left(\frac{t^2}{2} + C_1\right)\mathbf{i} + \left(e^t + C_2\right)\mathbf{j} + \left(e^t(t-1) + C_3\right)\mathbf{k}.
\end{aligned}$$

Since $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$, we have

$$\begin{cases} \frac{(1)^2}{2} + C_1 = 1 \\ e^{(1)} + C_2 = 1 \\ e^{(1)}((1) - 1) + C_3 = 1 \end{cases}.$$

These imply that $C_1 = \frac{1}{2}$, $C_2 = 1 - e$, and $C_3 = 1$. Thus, we have

$$\mathbf{r}(t) = \left(\frac{t^2}{2} + \frac{1}{2}\right)\mathbf{i} + \left(e^t + 1 - e\right)\mathbf{j} + \left(te^t - e^t + 1\right)\mathbf{k}.$$

■

10.2 #44 Find an expression for

$$\frac{d}{dt}\left[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t))\right].$$

[Solution]

$$\begin{aligned}
& \frac{d}{dt}\left[\mathbf{u}(t)\cdot(\mathbf{v}(t)\times\mathbf{w}(t))\right] \\
= & \left[\frac{d}{dt}\mathbf{u}(t)\right]\cdot(\mathbf{v}(t)\times\mathbf{w}(t))+\mathbf{u}(t)\cdot\frac{d}{dt}(\mathbf{v}(t)\times\mathbf{w}(t)) \\
= & \mathbf{u}'(t)\cdot(\mathbf{v}(t)\times\mathbf{w}(t))+\mathbf{u}(t)\cdot\left[\frac{d}{dt}\mathbf{v}(t)\times\mathbf{w}(t)+\mathbf{v}(t)\times\frac{d}{dt}\mathbf{w}(t)\right] \\
= & \mathbf{u}'(t)\cdot(\mathbf{v}(t)\times\mathbf{w}(t))+\mathbf{u}(t)\cdot(\mathbf{v}'(t)\times\mathbf{w}(t))+\mathbf{u}(t)\cdot(\mathbf{v}(t)\times\mathbf{w}'(t)).
\end{aligned}$$

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