1. Find the limit

\[ \lim_{(x,y) \to (0,0)} \frac{xy \cos y}{3x^2 + y^2}, \]

if it exists, or show that the limit does not exist.

[Solution]
Assume that \( x = 0 \). The limit becomes

\[ \lim_{(0,y) \to (0,0)} \frac{0 \cdot y \cos y}{3 \cdot 0^2 + y^2} = \lim_{(0,y) \to (0,0)} \frac{0}{y^2} = \lim_{(0,y) \to (0,0)} 0 = 0. \]

Assume that \( x = y \). The limit becomes

\[ \lim_{(x,x) \to (0,0)} \frac{x \cdot x \cos x}{3x^2 + x^2} = \lim_{(x,x) \to (0,0)} \frac{x^2 \cos x}{4x^2} = \lim_{(x,x) \to (0,0)} \frac{\cos x}{4} = \frac{1}{4}. \]

We got two different limits by approaching along two different paths. Thus, the limit does not exist.

2. Find the limit

\[ \lim_{(x,y) \to (0,0)} (x^2 + y^2) \ln (x^2 + y^2). \]

[Solution]
Note that if \((r, \theta)\) are polar coordinates of the point \((x, y)\) with \( r \geq 0 \), note that \( r \to 0^+ \) as \((x, y) \to (0, 0)\). The polar coordinates are \( x = r \cos \theta \) and \( y = r \sin \theta \). This implies that \( x^2 + y^2 = r^2 \). Thus, the limit becomes

\[ \lim_{(x,y) \to (0,0)} (x^2 + y^2) \ln (x^2 + y^2) \]

\[ = \lim_{r \to 0^+} r^2 \ln (r^2) = \lim_{r \to 0^+} r^2 2 \ln r \]

\[ = \lim_{r \to 0^+} 2 \ln r \lim_{r \to 0^+} \frac{1}{r} = 2 \lim_{r \to 0^+} \frac{1}{r} = -2 \lim_{r \to 0^+} r^3 = 0. \]

3. Find an equation of the tangent plane to the given parametric surface

\[ \mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + v\mathbf{k}. \]

to the point \((u,v) = (1,1)\).

[Solution]
There are two first partial derivatives.

\[ \mathbf{r}_u (u, v) = vi + j. \]

and

\[ \mathbf{r}_v (u, v) = ui + k. \]

At the point \((u, v) = (1, 1)\), we have

\[ \mathbf{r} (1, 1) = (1)(1) \mathbf{i} + (1) \mathbf{j} + (1) \mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \]

\[ \mathbf{r}_u (1, 1) = (1) \mathbf{i} + \mathbf{j} = \mathbf{i} + \mathbf{j}, \]

and

\[ \mathbf{r}_v (1, 1) = (1) \mathbf{i} + \mathbf{k} = \mathbf{i} + \mathbf{k}. \]

The normal vector at the point \((u, v) = (1, 1)\) is

\[ \mathbf{r}_u \times \mathbf{r}_v (1, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k} = \mathbf{i} - \mathbf{j} - \mathbf{k}. \]

So, the equation of the tangent plane is

\[ 1 (x - 1) - 1 (y - 1) - 1 (z - 1) = 0, \]

or,\[ x - y - z + 1 = 0. \]

4. Find \(\frac{dy}{dx}\) where

\[ \sin x + \cos y = \sin x \cos y. \]

[Solution]

Let

\[ F (x, y) = \sin x + \cos y - \sin x \cos y. \]

We have

\[ \frac{\partial F}{\partial x} = \cos x - \cos x \cos y \]

and

\[ \frac{\partial F}{\partial y} = - \sin y + \sin x \sin y. \]

Thus, by equation 6, we have

\[ \frac{dy}{dx} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = - \frac{\cos x - \cos x \cos y}{- \sin y + \sin x \sin y}. \]

5. Find \(\frac{dF}{dr}\) and \(\frac{dF}{dy}\) where

\[ F (x, y) = x^2 + 2x + y^2 \]

where \(x = r \cos \theta\) and \(y = r \sin \theta.\)

[Solution]
By the Chain Rule, we have
\[ \frac{dF}{dr} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial r} \]
\[ = (2x + 2) (\cos \theta) + (2y) (\sin \theta) \]
\[ = (2 (r \cos \theta) + 2) (\cos \theta) + (2 (r \sin \theta)) (\sin \theta) \]
\[ = 2r + 2 \cos \theta \]
and
\[ \frac{dF}{d\theta} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial \theta} \]
\[ = (2x + 2) (-r \sin \theta) + (2y) (r \cos \theta) \]
\[ = (2 (r \cos \theta) + 2) (-r \sin \theta) + (2 (r \sin \theta)) (r \cos \theta) \]
\[ = -2r \sin \theta. \]

6. Find the directional derivative of the function
\[ f(x, y) = e^{2y} \ln x \]
at the point \((1, 0, 0)\) in the direction of the vector \(\mathbf{v} = \langle 5, -12 \rangle\).

[Solution]
First, we need an unit vector of \(\mathbf{v}\). The unit vector
\[ \mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{(5, -12)}{\sqrt{(5)^2+(-12)^2}} = \left\langle \frac{5}{13}, \frac{-12}{13} \right\rangle. \]
Second, the gradient of \(f\) is
\[ \nabla f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left( \frac{e^{2y}}{x}, 2e^{2y} \ln x \right). \]
Thus, the directional derivative at the point \((1, 0, 0)\) is
\[ D_{\mathbf{u}} f(1, 0) = \nabla f(1, 0) \cdot \left\langle \frac{5}{13}, \frac{-12}{13} \right\rangle = (1, 0) \cdot \left\langle \frac{5}{13}, \frac{-12}{13} \right\rangle = \frac{5}{13}. \]

7. Find the critical points of the function
\[ f(x, y) = (x^2 - 4x) (2y - y^2). \]

[Solution]
First, we have
\[ \nabla f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left( (2x - 4) (2y - y^2), (x^2 - 4x) (2 - 2y) \right). \]
By assuming \(\nabla f(x, y) = 0\), we have
\[ \begin{cases} 
(2x - 4) (2y - y^2) = 0 \\
(x^2 - 4x) (2 - 2y) = 0 
\end{cases}. \]
From the first equation, we have \(2x - 4 = 0\), or \(2y - y^2 = 0\). This tells us that \(x = 2\), or, \(y = 0\) or \(2\). Similarly, From the second equation, we have \(x^2 - 4x = 0\), or \(2 - 2y = 0\). This tells us that \(x = 0\) or \(4\), or, \(y = 1\). There are several combinations:
(a) If \(x = 2\), then \(y\) must be \(1\) from the second equation.
(b) If \(y = 0\), then \(x\) can be \(0\) or \(4\) from the second equation.
(c) If \(y = 2\), then \(x\) can be \(0\) or \(4\) from the second equation.
Therefore, we have five critical points $(2, 1)$, $(0, 0)$, $(4, 0)$, $(0, 2)$ and $(4, 2)$.

8. Find the local maximum and minimum values and saddle point(s) of the function

$$f (x, y) = (x^2 - 4x) (2y - y^2).$$

**[Solution]**

From problem 7, we have five critical points $(2, 1)$, $(0, 0)$, $(4, 0)$, $(0, 4)$ and $(4, 4)$. To classify them, we need the determinant

$$\Delta (x, y) = f_{xx}f_{yy} - f_{xy}^2 = ((2 (2y - y^2)) ((x^2 - 4x) (-2)) - ((2x - 4) (2 - 2y))^2$$

$$= -4xy (2 - y) (x - 4) - 16 (x - 2)^2 (1 - y)^2$$

and $f_{xx} (x, y) = (2) (2y - y^2) = 2y (2 - y)$.

(a) For the critical point $(2, 1)$, $\Delta (2, 1) = 16 > 0$ and $f_{xx} (1, 1) = 2 > 0$. Thus, it is a local minimum.

(b) For the critical point $(0, 0)$, $\Delta (0, 0) = -64 < 0$. Thus, it is a saddle point.

(c) For the critical point $(4, 0)$, $\Delta (4, 0) = -64 < 0$. Thus, it is a saddle point.

(d) For the critical point $(0, 2)$, $\Delta (0, 2) = -64 < 0$. Thus, it is a saddle point.

(e) For the critical point $(4, 2)$, $\Delta (4, 2) = -64 < 0$. Thus, it is a saddle point.

9. Find the critical points of the function

$$f (x, y) = 2x^2 + 3y^2 - 4x - 5.$$

**[Solution]**

First, we calculate

$$\nabla f (x, y) = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right> = (4x - 4, 6y).$$

Assume that $\nabla f = 0$. This implies that $(x, y) = (1, 0)$.

10. Find the extreme values of

$$f (x, y) = 2x^2 + 3y^2 - 4x - 5$$

subject to the constraint

$$x^2 + y^2 = 16.$$

**[Solution]**

Let $g (x, y) = x^2 + y^2 - 16$. Thus, $g = 0$ becomes a constraint.

If $\nabla g (x, y) = 0$, then $(x, y) = (0, 0, 0)$ which does not satisfy the constraint $g = 0$.

Assume that $\nabla f = \lambda \nabla g$. With the constraint, we have

$$\begin{align*}
4x - 4 &= 2\lambda x \\
6y &= 2\lambda y \\
x^2 + y^2 &= 16
\end{align*}$$

If $y \neq 0$, then by the second equation, we have $\lambda = 3$. This implies that $x = -2$ and $y = \pm \sqrt{12}$. In this case, we have two critical points $(-2, 2\sqrt{3})$ and $(-2, -2\sqrt{3})$. The values are both $f (-2, 2\sqrt{3}) = f (-2, -2\sqrt{3}) = 47$.

If $y = 0$, then $x = \pm 4$. In this case, we have two critical points $(4, 0)$ and $(-4, 0)$. The values of them are $f (4, 0) = 11$ and $f (-4, 0) = 43$. 


Hence, the maximum is 47 at the points \((-2, 2\sqrt{3})\) and \((-2, -2\sqrt{3})\) and the minimum is 11 at the point \((4, 0)\).

11. Find the extreme values of
\[
f(x, y) = 2x^2 + 3y^2 - 4x - 5
\]
on the region described by
\[
x^2 + y^2 \leq 16.
\]

[Solution]
Let \(g(x, y) = x^2 + y^2 - 16\). First, we calculate
\[
\nabla f (x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \langle 4x - 4, 6y \rangle
\]
and
\[
\nabla g (x, y) = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{pmatrix} = \langle 2x, 2y \rangle.
\]

First, we find the extreme values for the interior of \(x^2 + y^2 \leq 16\). Consider the region \(x^2 + y^2 < 16\). Assume that \(\nabla f = 0\). This implies that \((x, y) = (1, 0)\). Since \((1, 0)\) satisfies \(x^2 + y^2 < 16\), \((1, 0)\) is a critical point. The value is \(f(1, 0) = -7\).

Next, we consider the boundary \(x^2 + y^2 = 16\). Thus, \(g = 0\) becomes a constraint. If \(\nabla g (x, y) = 0\), then \((x, y) = (0, 0, 0)\) which does not satisfy the constraint \(g = 0\).
Assume that \(\nabla f = \lambda \nabla g\). With the constraint, we have
\[
\begin{cases}
4x - 4 &= 2\lambda x \\
6y &= 2\lambda y \\
x^2 + y^2 &= 16
\end{cases}
\]
If \(y \neq 0\), then by the second equation, we have \(\lambda = 3\). This implies that \(x = -2\) and \(y = \pm \sqrt{12}\). In this case, we have two critical points \((-2, 2\sqrt{3})\) and \((-2, -2\sqrt{3})\). The values are both \(f(-2, 2\sqrt{3}) = f(-2, -2\sqrt{3}) = 47\).

If \(y = 0\), then \(x = \pm 4\). In this case, we have two critical points \((4, 0)\) and \((-4, 0)\). The values of them are \(f(4, 0) = 11\) and \(f(-4, 0) = 43\).
Hence, the maximum is 47 at the points \((-2, 2\sqrt{3})\) and \((-2, -2\sqrt{3})\) and the minimum is \(-7\) at the point \((1, 0)\).

12. Evaluate the double integral
\[
\iint_D (x + y) \, dA,
\]
where \(D\) is bounded by \(y = \sqrt{x}\) and \(y = x^2\).

[Solution]
Set $\sqrt{x} = x^3$. We can solve for $x = 1$ or 0. Thus,

$$
\int_0^1 \int_{\sqrt{x}}^x (x + y) \, dy \, dx = \int_0^1 \left[ \int_{\sqrt{x}}^x (x + y) \, dy \right] \, dx
$$

$$
= \int_0^1 \left[ \left( xy + \frac{y^2}{2} \right) \bigg|_{y=\sqrt{x}} \right] \, dx
$$

$$
= \int_0^1 \left[ \left( x(\sqrt{x}) + \frac{\sqrt{x})^2}{2} \right) - \left( x(\sqrt{x}) + \frac{\sqrt{x})^2}{2} \right) \right] \, dx
$$

$$
= \int_0^1 \left[ x^2 + \frac{x}{2} - x^3 - \frac{x^4}{2} \right] \, dx
$$

$$
= \left( \frac{2}{5}x^\frac{5}{2} + \frac{x^2}{4} - \frac{x^4}{4} - \frac{x^5}{10} \right) \bigg|_{x=0}^{x=1}
$$

$$
= \left( \frac{2}{5}(1)^\frac{5}{2} + \frac{(1)^2}{4} - \frac{(1)^4}{4} - \frac{(1)^5}{10} \right) - \left( \frac{2}{5}(0)^\frac{5}{2} + \frac{(0)^2}{4} - \frac{(0)^4}{4} - \frac{(0)^5}{10} \right)
$$

$$
= \frac{3}{10}.
$$

13. Evaluate the integral:

$$
\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2} \, dx \, dy.
$$

[Solution]

According to the region, by interchanging the order, we have

$$
\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2} \, dx \, dy = \int_0^1 \int_0^{2x} e^{x^2} \, dy \, dx
$$

$$
= \int_0^1 e^{x^2} \left[ (y) \bigg|_{y=0}^{y=2x} \right] \, dx
$$

$$
= \int_0^1 e^{x^2} \left[ (2x) - (0) \right] \, dx
$$

$$
= \int_0^1 2xe^{x^2} \, dx
$$

$$
= \left[ \frac{e^{x^2}}{2} \right]_{x=0}^{x=1}
$$

$$
= e^{\frac{1}{2}} - e^0
$$

$$
= e - 1.
$$
14. Evaluate
\[ \iint_D (x^2 + y^2)^{\frac{3}{2}} \, dA \]
where \( D \) is the half disk \( x^2 + y^2 \leq 4 \) with \( x \geq 0 \).

**[Solution]**
Using polar coordinates, let \( x = r \cos \theta, y = r \sin \theta \) where \( 0 \leq r \leq 2 \) and \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \).
The reason \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\) is for right-half disk. Therefore,
\[
\iint_D (x^2 + y^2)^{\frac{3}{2}} \, dA
= \int_0^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [(r \cos \theta)^2 + (r \sin \theta)^2]^{\frac{3}{2}} \cdot |r| \, d\theta \, dr
= \int_0^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^3 \, d\theta \, dr
= \left( \int_0^2 r^3 \, dr \right) \cdot \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \right)
= \left( \frac{r^4}{9} \right|_{r=0}^{r=2} \cdot \left( \theta \big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \right)
= \left( \frac{2^4}{9} - 0 \right) \cdot \left( \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right)
= \frac{512}{9} \pi.
\]

15. Find the volume of the solid that lies under the plane \( x - 2y + z = 4 \) and above the square \( R = [-1, 1] \times [0, 2] \).

**[Solution]**
The volume is
\[
\iint_R (4 - x + 2y) \, dA
= \int_{-1}^{1} \int_{0}^{2} (4 - x + 2y) \, dy \, dx
= \int_{-1}^{1} \left[ \int_{0}^{2} (4 - x + 2y) \, dy \right] \, dx
= \int_{-1}^{1} \left[ 4y - xy + y^2 \bigg|_{y=0}^{y=2} \right] \, dx
= \int_{-1}^{1} \left[ (4(2) - x(2) + 2^2) - (4(0) - x(0) + 0^2) \right] \, dx
= \int_{-1}^{1} (12 - 2x) \, dx
= (12x + x^2) \bigg|_{x=-1}^{x=1} = (12(1) + (1)^2) - (12(-1) + (-1)^2)
= 24.
\]