1. (10pts) (9.1. #16) Find an equation of a sphere if one of its diameters has end-points (2, 1, 4) and (4, 3, 10).

[Solution]
Since this diameter has end-points (2, 1, 4) and (4, 3, 10), the midpoint of the line segment with endpoints (2, 1, 4) and (4, 3, 10) is the center of this sphere and the radius is the half of the length of this line segment.

The midpoint is \((\frac{2+4}{2}, \frac{1+3}{2}, \frac{4+10}{2}) = (3, 2, 7)\). The length of the line segment is

\[
\sqrt{(4 - 2)^2 + (3 - 1)^2 + (10 - 4)^2} = 2\sqrt{11}.
\]

So, the radius is \(\sqrt{11}\). Therefore, the equation is

\[
(x - 3)^2 + (x - 2)^2 + (z - 7)^2 = \left(\sqrt{11}\right)^2 = 11.
\]

2. (10pts) (9.2. #32) Suppose a vector \(\mathbf{a}\) makes angles \(\alpha\), \(\beta\) and \(\gamma\) with the positive \(x\)-, \(y\)-, and \(z\)-axes, respectively. Find the components of \(\mathbf{a}\) and show that

\[
\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.
\]

[Solution]
Consider the vector \(\mathbf{a}\) and the \(x\)-axe. We can draw a line from the end point of the vector \(\mathbf{a}\) to the \(x\)-axe which forms a right angle with the \(x\)-axe. Now, you have a right-angle triangle with the hypotenuse is \(\mathbf{a}\) and the adjacent side is on the \(x\)-axe with the angle \(\alpha\) in between these two sides. So, the length of the adjacent side can be calculated by \(|\mathbf{a}|\cos\alpha\). This is the \(x\)-coordinate of \(\mathbf{a}\). Similarly, we can find the \(y\)- and \(z\)-coordinates of \(\mathbf{a}\). We conclude that \(\mathbf{a} = (|\mathbf{a}|\cos\alpha, |\mathbf{a}|\cos\beta, |\mathbf{a}|\cos\gamma)\).

The length of \(\mathbf{a}\) is

\[
|\mathbf{a}| = \sqrt{(|\mathbf{a}|\cos\alpha)^2 + (|\mathbf{a}|\cos\beta)^2 + (|\mathbf{a}|\cos\gamma)^2}.
\]

This implies that

\[
|\mathbf{a}|^2 = (|\mathbf{a}|\cos\alpha)^2 + (|\mathbf{a}|\cos\beta)^2 + (|\mathbf{a}|\cos\gamma)^2
\]

\[
\Rightarrow |\mathbf{a}|^2 = |\mathbf{a}|^2\cos^2\alpha + |\mathbf{a}|^2\cos^2\beta + |\mathbf{a}|^2\cos^2\gamma
\]

\[
\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.
\]
3. (10pts) (9.3. #28) For the vectors \( \mathbf{a} = (1, 2) \) and \( \mathbf{b} = (-4, 1) \), find \( \text{orth}_a \mathbf{b} \).

[Solution]
From exercise 27, we have \( \text{orth}_a \mathbf{b} = \mathbf{b} - \text{proj}_a \mathbf{b} \). By the formula in the textbook page 655,

\[
\text{proj}_a \mathbf{b} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||^2} \right) \mathbf{a} = \left( \frac{(1)(-4) + (2)(1)}{1^2 + 2^2} \right) (1, 2) = \left( \frac{-2}{5} \right) (1, 2) = \left( \frac{-2}{5}, \frac{-4}{5} \right).
\]

Thus,

\[
\text{orth}_a \mathbf{b} = \mathbf{b} - \text{proj}_a \mathbf{b} = (-4, 1) - \left( \frac{-2}{5}, \frac{-4}{5} \right) = \left( \frac{-18}{5}, \frac{9}{5} \right).
\]

4. (10pts) (9.4. #16) Find the area of the parallelogram with vertices \( K(1, 2, 3), L(1, 3, 6), M(3, 8, 6) \), and \( N(3, 7, 3) \).

[Solution]
After plotting these four points in a coordinate system, we have a parallelogram \( KLMN \). The vector \( \overrightarrow{KL} = (1 - 1, 3 - 2, 6 - 3) = (0, 1, 3) \) and \( \overrightarrow{KN} = (3 - 1, 7 - 2, 3 - 3) = (2, 5, 0) \). Thus, the area of the parallelogram \( KLMN \) is

\[
|\overrightarrow{KL} \times \overrightarrow{KN}| = \begin{vmatrix} i & j & k \\ 0 & 1 & 3 \\ 2 & 5 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} i - 0 \begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} j + 0 \begin{vmatrix} 0 & 1 \\ 2 & 5 \end{vmatrix} k = (-15i + 6j - 2k) = \sqrt{(-15)^2 + (6)^2 + (-2)^2} = \sqrt{265}.
\]

5. (10pts) (9.5. #10) Find the parametric equation and symmetric equation for the line of intersection of the planes \( x + y + z = 1 \) and \( x + z = 0 \).

[Solution]
To write down a line equation, we need a directional vector and a point. To get a point, first, we assume that \( z = 0 \). So, we have \( x + y = 1 \) and \( x = 0 \). This implies that \( x = 0 \) and \( y = 1 \). Thus, the point \((0, 1, 0)\) is in the intersection of our two planes, which is a point we need.

We can read the normal vector of the plane \( x + y + z = 1 \) to be \((1, 1, 1)\) and the normal vector of the plane \( x + z = 0 \) to be \((1, 0, 1)\). Since the intersection line lies on both planes,
the line must be perpendicular to both normal vectors. So, the directional vector of the
line can be \((1,1,1) \times (1,0,1) = (1,0,-1)\).

Therefore, the parametric equation of the line is
\[
\begin{align*}
  x &= 0 + (1) t = t \\
  y &= 1 + (0) t = 1 \\
  z &= 0 + (-1) t = -t
\end{align*}
\]
and the symmetric equation of the line should look like
\[
\frac{x - 0}{1} = \frac{y - 1}{0} = \frac{z - 0}{-1}.
\]
But, we cannot divide by 0. Thus, we should write a equation for \(y\) individually. So, the
symmetric equation of the line is
\[
\frac{x - 0}{1} = \frac{z - 0}{-1} \text{ and } y = 1.
\]

6. (10pts) (9.5. #30) Find the equation of the plane that passes through the line of intersection of
the planes \(x - z = 1\) and \(y + 2z = 3\) and is perpendicular to the plane \(x + y - 2z = 1\).

[Solution]
We can read the normal vector of the plane \(x - z = 1\) is \((1,0,-1)\) and the normal
vector of the plane \(y + 2z = 3\) is \((0,1,2)\). The directional vector of the line of intersection
of both planes is perpendicular to both. So, it can be \((1,0,-1) \times (0,1,2) = (1,-2,1)\).
Also, by letting \(z = 0\) for both planes, we can have a point in the line of intersection
\((1,3,0)\).

The normal vector of the plane \(x + y - 2z = 1\) is \((1,1,-2)\). Since our plane is perpen-
dicular to the plane \(x + y - 2z = 1\), we know that \((1,1,-2)\) is on our plane.

Now, we have two vectors \((1,-2,1)\) and \((1,1,-2)\) on our plane and a point \((1,3,0)\).
Thus we have the normal vector of our plane is \((1,-2,1) \times (1,1,-2) = (3,3,3)\). Hence,
the equation of the plane is
\[
(x - 1, y - 3, z - 0) \cdot (3,3,3) = 0,
\]
which is
\[
x + y + z = 4.
\]

7. (10pts) (9.7. #20) Identify the surface \(r^2 - 2z^2 = 4\).

[Solution]
We see \(r\) and \(z\). It suggests that we are using cylindrical coordinates.
In cylindrical coordinates, we have \(r^2 = x^2 + y^2\). Thus, our equation becomes \(x^2 + y^2 - 2z^2 = 4\). By dividing by 4, we get
\[
\frac{x^2}{4} + \frac{y^2}{4} - \frac{z^2}{2} = 1.
\]
By looking at Table 2 in page 682, we identify this surface as a hyperboloid of one sheet.

8. (10pts) (10.2. #20) Find parametric equations for the tangent line to the curve \(x = 2 \cos t,\)
\(y = 2 \sin t,\) and \(z = 4 \cos 2t\) at the point \((\sqrt{3},1,2)\). Illustrate by graphing both the curve
and the tangent line on a common screen.

[Solution]
Let the curve be represented by a vector function \( \mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 4 \cos 2t \rangle \).

When \( t = \frac{\pi}{6} \), we have the point \((\sqrt{3}, 1, 2)\). So, the directional vector of the tangent line at \((\sqrt{3}, 1, 2)\) is

\[
\mathbf{r}' \left( \frac{\pi}{6} \right) = \langle -2 \sin t, 2 \cos t, -8 \sin 2t \rangle|_{t=\frac{\pi}{6}} = \langle -1, \sqrt{3}, -4\sqrt{3} \rangle.
\]

Thus, the parametric equations for the tangent line are

\[
\begin{align*}
x &= \sqrt{3} + (-1)t = \sqrt{3} - t \\
y &= 1 + (\sqrt{3})t = 1 + \sqrt{3}t \\
z &= 2 + (-4\sqrt{3})t = 2 - 4\sqrt{3}t
\end{align*}
\]

9. (10pts) (10.3. #39) Find the vector \( \mathbf{T} \), \( \mathbf{N} \), \( \mathbf{B} \), and the curvature \( \kappa \) of \( \mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle \) at the point \((1, \frac{2}{3}, 1)\).

[Solution]

We have

\[
\mathbf{r}'(t) = \langle 2t, 2t^2, 1 \rangle.
\]

This implies that

\[
|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (2t^2)^2 + 1^2} = \sqrt{4t^2 + 4t^4 + 1} = \sqrt{(2t + 1)^2} = 2t + 1.
\]

Thus,

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \langle \frac{2t}{2t + 1}, \frac{2t^2}{2t + 1}, \frac{1}{2t + 1} \rangle.
\]

This implies that

\[
\mathbf{N}(1) = \frac{\mathbf{T}'(1)}{|\mathbf{T}'(1)|} = \frac{\langle 2 \frac{8}{9}, 8, -2 \rangle}{\sqrt{(\frac{2}{9})^2 + (\frac{8}{9})^2 + (-\frac{2}{9})^2}} = \frac{3}{2\sqrt{2}} \langle 2 \frac{8}{9}, 8, -2 \rangle
\]

\[
= \left\langle \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}} \right\rangle.
\]

Therefore,

\[
\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{2}{3} & \frac{2}{3} & 1 \\
\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{vmatrix}
= -\frac{\sqrt{2}}{3}\mathbf{i} + \frac{\sqrt{2}}{6}\mathbf{j} + \frac{\sqrt{2}}{3}\mathbf{k}.
\]

Finally, when \( t = 1 \),

\[
\kappa = \frac{|\mathbf{T}'(1)|}{|\mathbf{r}'(1)|} = \frac{\sqrt{(\frac{2}{9})^2 + (\frac{8}{9})^2 + (-\frac{2}{9})^2}}{2(1 + 1)} = \frac{2\sqrt{2}}{9}.
\]
10. (10pts) (10.5. #19) Find a parametric representation for the part of the hyperboloid \( x^2 + y^2 - z^2 = 1 \) that lies to the right of the \( xz \)-plane.

**[Solution]**

In hyperboloid, with a fixed \( z \), we have a circle. This suggests us that we can set \( x = r \cos \theta \) and \( y = r \sin \theta \), where \( 0 \leq \theta \leq 2\pi \) and \( r \geq 0 \). Thus, \( z^2 = 1 - x^2 - y^2 = 1 - r^2 \cos^2 \theta - r^2 \sin^2 \theta = 1 - r^2 \). The parametric representation is

\[
\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, 1 - r^2 \rangle.
\]

Since it lies on the right of the \( xz \)-plane, it means that \( y \geq 0 \). This tells us that \( r \sin \theta \geq 0 \). Since \( r \geq 0 \), we have \( \sin \theta \geq 0 \). It turns out that \( 0 \leq \theta \leq \pi \). Therefore, we have the parametric representation

\[
\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, 1 - r^2 \rangle
\]

where \( r \geq 0 \) and \( 0 \leq \theta \leq \pi \).