THE LARGE SCALE STRUCTURE
OF SPACE–TIME

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5.6   The Kerr solution

In general, astronomical bodies are rotating and so one would not expect the solution outside them to be exactly spherically symmetric. The Kerr solutions are the only known family of exact solutions which could represent the stationary axisymmetric asymptotically flat field outside a rotating massive object. They will be the exterior solutions only for massive rotating bodies with a particular combination of multipole moments; bodies with different combinations of moments will have other exterior solutions. The Kerr solutions do however appear to be the only possible exterior solutions for black holes (see § 9.2 and § 9.3).

The solutions can be given in Boyer and Lindquist coordinates (r, θ, ϕ, t) in which the metric takes the form

$$ds^2 = \rho^2 \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + (r^2 + a^2) \sin^2 \theta \, d\phi^2 - dt^2 + \frac{2mr}{\rho^2} (a \sin^2 \theta \, d\phi - dt)^2,$$

$$(5.29)$$

where $\rho^2(r, \theta) \equiv r^2 + a^2 \cos^2 \theta$ and $\Delta(r) \equiv r^2 - 2mr + a^2.$

m and a are constants, m representing the mass and ma the angular momentum as measured from infinity (Boyer and Price (1965)); when $a = 0$ the solution reduces to the Schwarzschild solution. This metric form is clearly invariant under simultaneous inversion of t and ϕ, i.e. under the transformation $t \to -t$, $\phi \to -\phi$, although it is not invariant under inversion of t alone (except when $a = 0$). This is what one would expect, since time inversion of a rotating object produces an object rotating in the opposite direction.

When $a^2 > m^2$, $\Delta > 0$ and the above metric is singular only when $r = 0$. The singularity at $r = 0$ is not in fact a point but a ring, as can be seen by transforming to Kerr–Schild coordinates $(x, y, z, \bar{t})$, where

$$x + iy = (r + ia) \sin\theta \exp i \int (d\phi + a\Delta^{-1} dr),$$

$$z = r\cos\theta, \quad \bar{t} = \int (dt + (r^2 + a^2)\Delta^{-1} dr) - r.$$

In these coordinates, the metric takes the form

$$ds^2 = dx^2 + dy^2 + dz^2 - d\bar{t}^2$$

$$+ \frac{2mr^3}{r^4 + a^2 z^2} \left(\frac{r(x\,dx + y\,dy) - a(x\,dy - y\,dx)}{r^2 + a^2} + \frac{z\,dz}{r} + d\bar{t} \right)^2, \quad (5.30)$$

where r is determined implicitly, up to a sign, in terms of x, y, z by

$$r^4 - (x^2 + y^2 + z^2 - a^2) r^2 - a^2 z^2 = 0.$$

For $r \neq 0$, the surfaces $\{r = \text{constant}\}$ are confocal ellipsoids in the (x, y, z) plane, which degenerate for $r = 0$ to the disc $z^2 + y^2 \leqslant a^2$, $z = 0$. The ring $x^2 + y^2 = a^2$, $z = 0$ which is the boundary of this disc, is a real curvature singularity as the scalar polynomial $R_{abcd} R^{abcd}$ diverges there. However no scalar polynomial diverges on the disc except at the boundary ring. The function r can in fact be analytically continued from positive to negative values through the interior of the disc $x^2 + y^2 < a^2$, $z = 0$, to obtain a maximal analytic extension of the solution.

To do this, one attaches another plane defined by coordinates (x', y', z') where a point on the top side of the disc $x^2 + y^2 < a^2$, $z = 0$ in the (x, y, z) plane is identified with a point with the same x and y coordinates on the bottom side of the corresponding disc in the (x', y', z') plane. Similarly a point on the bottom side of the disc in the (x, y, z) plane is identified with a point on the top side of the disc in the (x', y', z') plane (see figure 27). The metric (5.30) extends in the obvious way to this larger manifold. The metric on the (x', y', z') region is again of the form (5.29), but with negative rather than positive values of r. At large negative values of r, the space is again asymptotically flat but this time with negative mass. For small negative values of r near the ring singularity, the vector $\partial/\partial\phi$ is timelike, so the circles $(t = \text{constant}, r = \text{constant}, \theta = \text{constant})$ are closed timelike curves. These closed timelike curves can be deformed to pass through any point of the extended space (Carter (1968a)). This solution is geodesic-

ally incomplete at the ring singularity. However the only timelike and null geodesics which reach this singularity are those in the equatorial plane on the positive r side (Carter (1968a)).
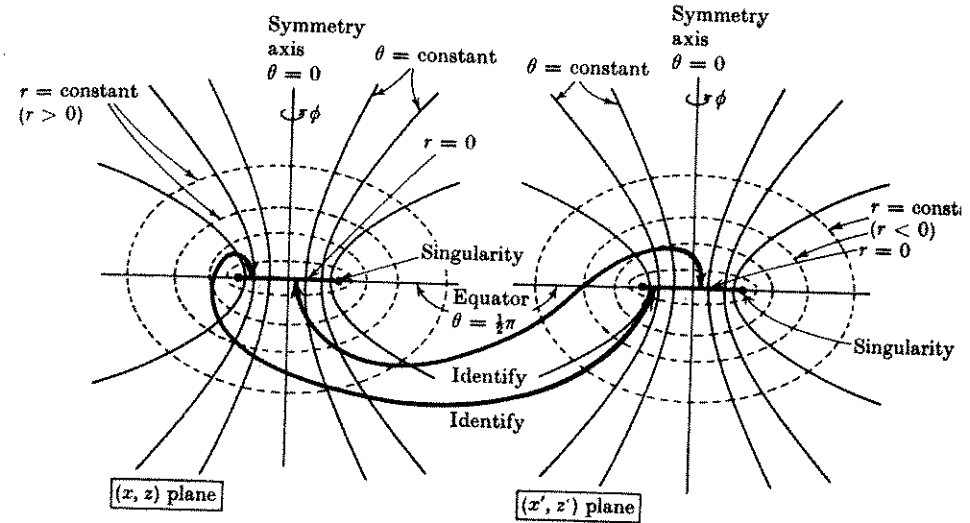

FIGURE 27. The maximal extension of the Kerr solution for $a^2 > m^2$ is obtained by identifying the top of the disc $x^2 + y^2 < a^2$, $z = 0$ in the (x, y, z) plane with the bottom of the corresponding disc in the (x', y', z') plane, and vice versa. The figure shows the sections $y = 0$, $y' = 0$ of these planes. On circling twice round the singularity at $x^2 + y^2 = a^2$, $z = 0$ one passes from the (x, y, z) plane to the (x', y', z') plane (where r is negative) and back to the (x, y, z) plane (where r is positive).

The extension in the case $a^2 < m^2$ is rather more complicated, because of the existence of the two values $r_+ = m + (m^2 - a^2)^{\frac{1}{2}}$ and $r_- = m - (m^2 - a^2)^{\frac{1}{2}}$ of r at which $\Delta(r)$ vanishes. These surfaces are similar to the surfaces $r = r_+$, $r = r_-$ in the Reissner–Nordström solution. To extend the metric across these surfaces, one transforms to the Kerr coordinates (r, θ, ϕ_+, u_+), where

$$du_+ = dt + (r^2 + a^2)\Delta^{-1} dr, \quad d\phi_+ = d\phi + a\Delta^{-1} dr.$$

The metric then takes the form

$$ds^2 = \rho^2\,d\theta^2 - 2a\sin^2\theta\,dr\,d\phi_+ + 2\,dr\,du_+$$

$$+ \rho^{-2}[(r^2 + a^2)^2 - \Delta a^2\sin^2\theta]\sin^2\theta\,d\phi_+{}^2$$

$$- 4a\rho^{-2}mr\sin^2\theta\,d\phi_+\,du_+ - (1 - 2mr\rho^{-2})\,du_+{}^2 \quad (5.31)$$
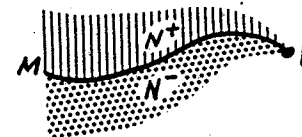
Knots & Links
D. Rolfsen
Ch 5

C. CONSTRUCTION OF THE CYCLIC COVERINGS OF A KNOT COMPLEMENT USING SEIFERT SURFACES.

There is an important class of covering spaces of a knot complement $X = S^{n+2} - K^n$, which will be used in the next chapter to define certain abelian invariants of K. Readers unfamiliar with covering space theory will find a synopsis in Appendix A.

Seifert surfaces give a convenient means of constructing these covering spaces, in a manner entirely analogous to 'cuts' in the classical theory of Riemann surfaces.

Let M^{n+1} be a Seifert surface for the knot K^n in S^{n+2} and let $N : \overset{\circ}{M} \times (-1,1) \to S^{n+2}$ be an open bicollar of the _interior_ $\overset{\circ}{M} = M - K$, so $\overset{\circ}{M} = N(\overset{\circ}{M} \times 0)$. We denote:

$$N = N(\overset{\circ}{M} \times (-1,1))$$
$$N^+ = N(\overset{\circ}{M} \times (0,1))$$
$$N^- = N(\overset{\circ}{M} \times (-1,0))$$
$$Y = S^{n+2} - M$$
$$X = S^{n+2} - K$$

Thus we have two triples (N, N^+, N^-) and (Y, N^+, N^-). Form countably many copies of each, denoted (N_i, N_i^+, N_i^-) and (Y_i, N_i^+, N_i^-), $i = 0, \pm 1, \pm 2, \cdots$. Let $\tilde{N} = \bigcup_{i=-\infty}^{\infty} N_i$ and $\tilde{Y} = \bigcup_{i=-\infty}^{\infty} Y_i$ be the disjoint unions. Finally, form an identification space by identifying $N_i^+ \subset Y_i$ with $N_i^+ \subset N_i$ via t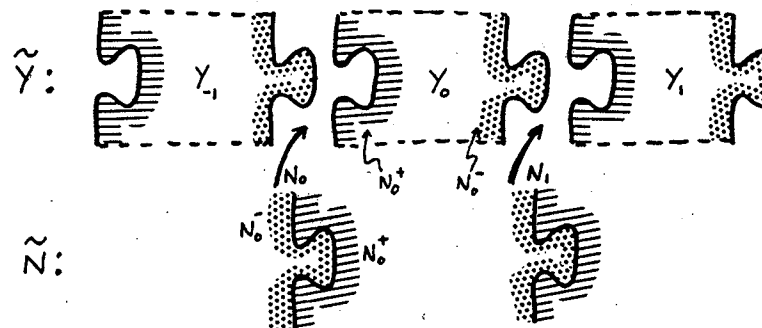he identity homeomorphism, and likewise identify each $N_i^- \subset Y_i$ with $N_{i+1}^- \subset N_{i+1}$. Call the resulting space $\tilde{X}$.

_1._ EXERCISE. Verify the following facts. $\tilde{X}$ is a path-connected open $(n+2)$ - manifold. There is a map $p : \tilde{X} \to X$ which is a regular covering

space. There is a covering automorphism $\tau : \tilde{X} \to \tilde{X}$, which takes Y_i to Y_{i+1} and N_i to N_{i+1}, and τ generates the group $\text{Aut}(\tilde{X})$, which is infinite cyclic.

2. DEFINITION. $\tilde{X}$ is called the __infinite cyclic__ cover of X .

3. PROPOSITION. $\tilde{X}$ is the universal abelian cover of X .

4. COROLLARY. $\tilde{X}$ depends (up to covering isomorphism) only on the knot type of K, and not on the choice of Seifert surface or other choices in the above construction.

PROOF OF THE PROPOSITION. We need only the fact that $\text{Aut}(\tilde{X}) \approx Z$. The exact sequence

$$1 \longrightarrow \pi_1(\tilde{X}) \xrightarrow{P_*} \pi_1(X) \longrightarrow \overset{Z}{\underset{\text{\rotatebox{90}{=}}}{\text{Aut}(\tilde{X})}} \longrightarrow 1$$

shows that $P_*\pi_1(\tilde{X})$ contains the commutator subgroup C of $\pi_1(X)$. Now the induced map

$$Z \approx \frac{\pi_1(X)}{C} \longrightarrow \text{Aut}(\tilde{X}) \approx Z$$

has kernel $\dfrac{P_*\pi_1(\tilde{X})}{C}$, and (being surjective) must be an isomorphism. Hence $P_*\pi_1(\tilde{X}) \approx C$ and the proposition follows.

5. REMARK. This construction can be streamlined by eliminating reference to the N_i which glue the Y_i together. Let $\bar{Y}_i$ denote the closure of Y_i in $\tilde{X}$.

6. EXERCISE : $\bar{Y}_i$ is homeomorphic with $S^{n+2} - (K \; N)$. This latter space might be called "$S^{n+2} - K$ cut open along M"; it has two boundary components.

each homeomorphic with $\overset{o}{M}$, and we may regard $\tilde{X}$ as the union of copies of these cut open spaces, suitably sewn together.

A similar construction yields the __finite cyclic covering__ __spaces__ of $X = S^{n+2} - K^n$. Choose a fixed integer $k > 1$ and consider copies (N_i, N_i^+, N_i^-) and (Y_i, N_i^+, N_i^-) as above, $i = 0, 1, \ldots, k-1$. Let

$$\tilde{Y} = \bigcup_{i=0}^{k-1} Y_i \quad \text{and} \quad \tilde{N} = \bigcup_{i=0}^{k-1} N_i \quad \text{and make}$$
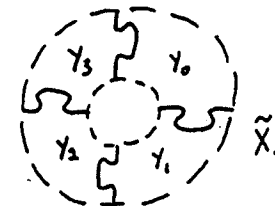
the same identifications as before, except that $N_{k-1}^- \subset Y_{k-1}$ is identified with $N_0^- \subset N_0$. Call the resulting space $\tilde{X}_k$. It is a k-fold 'cyclic' cover of X with $\text{Aut}(\tilde{X}) \approx Z/k$.

7. EXERCISE. Show that $\tilde{X}$ covers each $\tilde{X}_k$, so that $\tilde{X}_k$ may be regarded as a quotient space of $\tilde{X}$.

8. EXERCISE. Prove that $\tilde{X}_k$ corresponds to the kernel of the composite homomorphism

$$\pi_1(X) \to \frac{\pi_1(X)}{C} \approx Z \to Z/k$$

where $C < \pi_1(X)$ is the commutator subgroup and the right-hand map is the canonical projection.

9. COROLLARY. $\tilde{X}_k$ depends only on k and the knot type of K .