110:615 **algebraic topology I**, Fall 2016

Topology is the newest branch of mathematics. It originated around the turn of the twentieth century in response to Cantor, though its roots go back to Euler; it stands between algebra and analysis, and has had profound effects on both.

Since the 1950s topology has been at the cutting edge of mathematical research: its techniques have revolutionized algebraic geometry, number theory, physics (eg of condensed matter, not to mention string theory), as well as important parts (elliptic PDEs) of analysis. Since about 2000 it has been a significant source of ideas for the analysis of large structured data sets, and lately (via HoTT = higher order type theory) it has led to a rethinking of the foundations of logic and the philosophy of mathematics.

Topologists classify space for a living: for example, the three-dimensional space of politicians, as a subset of the five-dimensional manifold of real human beings. Knots and braids provide another class of interesting spaces, as does our eleven-dimensional (or so they would have us believe) own physical Universe. Phylogenetic trees in evolutionary theory are another class of examples just beginning to be studied.

615 is an introduction to algebraic topology as a way of thinking: not only in terms of its techniques, but as an opportunity to introduce a rich supply of concrete, and perhaps surprising, examples.

Draft schedule for 110.615 **classical algebraic topology**

It is a truth universally acknowledged, that there is no really satisfactory introductory algebraic topology textbook. This course attempts to verify this by providing a new example, cherrypicked from the best parts of several quite good standard choices.

A draft schedule follows. The course starts with a review of background material from geometry, which will be used as test examples throughout the course. The semester ends with the Poincaré duality theorem for manifolds; it is intended to lead into a second course centered around the model categorical approach to homotopy theory and homological algebra.

Please contact me ⟨jack@math.jhu.edu⟩ if you are interested, or have any questions. A rough draft for the course lectures is attached; but see the note at the end, about the first couple of weeks of classes.

— Week VI: (10 October)

3.8 examples: invariance of dimension, degree of a map, the orientation sheaf, the class of a submanifold, attaching a cell, p 59

Part III **Complexes and chains**

4.1 (abstract) simplicial complexes, eg Rips complexes, partition posets; simplicial chains, p 68
4.2 geometric realization, p 72
4.3 barycentric subdivision, p 74

— Week VII: (17 October)

4.4 products, p 76
4.5 simplicial sets; the classifying space of a category, BG and homotopy quotients, p 77

Basic homological algebra and verification of the axioms

5.1 chain complexes, chain homomorphisms, and chain homotopies, p 81

— Week VIII: (24 October)

5.2 singular homology; the homotopy axiom, p 88
5.3 locality of the singular complex, p 90
5.4 the snake lemma and the boundary homomorphism, p 96: Movietime!

https://www.youtube.com/watch?v=etbcKWEKnvg

— Week IX (31 October)

Part IV **Back to geometry!**

The stable homotopy category of finite cell complexes

6.1 cell complexes; the homotopy type of a cell complex, cellular chains, p 98
6.2 uniqueness of homology. Statement (not proof!) of theorems of Whitehead and Kan, p 102
6.3 sketch of the stable homotopy category, versus the homotopy category of chain complexes. Naive definition of naive spectra; statement (not proof) of Brown's representability theorem, p 105

— Week X (7 November)

COHOMOLOGY

7.1 Definition, axioms for the algebra and module structures; the Alexander-Whitney map, p 112
7.2 examples, p 117
7.2 cap products; the Eilenberg-Zilber map; the Künneth theorem foreshadowed, p 120

— Week XI (14 November)

POINCARÉ DUALITY

8.1 introduction, p 129
8.1 The orientation class, p 125
8.2 proof of the theorem, p 128

— Week XII (28 November)

8.4 applications: Intersection theory and Lefschetz' theorem. The Pontryagin-Thom collapse map, the Thom isomorphism theorem; bivariant functors, p 131

— Week XIII (5 December): **Margin for error!**

[**Appendices:** (to appear?)

On π_1: van Kampen, Hurewicz; Reidemeister moves and braid groups; Wirtinger's presentation of $\pi_1(S^3 - k)$; skein relations and the Alexander polynomial; covering spaces, eg of surfaces and configuration spaces; twisted coefficients; Chern classes, eg of line bundles; elliptic curves and

$$1 \to \mathbb{Z} \to \mathrm{Br}_3 \to \mathrm{Sl}_2(\mathbb{Z}) \to 1 .$$

deRham cohomology: Poincaré's lemma; the Hodge operator and duality; Maxwell's equations]

ABOUT THE FIRST COUPLE OF WEEKS:

If you are interested in the class, please **contact me**, and I will send you a set of class notes, compiled from previous years of teaching this course. [The file is too big to attach here.] The first fifteen pages or so are mostly review (spaces and functions, abelian groups . . .) and, rather than go over that in detail, I'd rather spend our class time on examples. I've attached few pages concerned with such examples after this.

Some references:

R Ghrist, **Elementary applied topology** (2014)

M Greenberg, **Lectures on algebraic topology** (1967)

J Rotman, **An introduction to algebraic topology** (1988)

A Hatcher, **Algebraic topology** (2002)

Politicians' uniquely simple personalities

The complexity of human personality has been reduced to five dimensions, based on factor analyses of judgements of personality traits[1]. Many researchers agree that a five-factor model of personality captures the essential features of all traits that are used to describe personality: energy/extroversion; agreeableness/friendliness; conscientiousness; emotional stability against neuroticism; and intellect/openness to experience[2–4]. But we show here that this common, standard set of five factors does not hold for judgements of famous political figures.

We found that, when people judge the personality traits of politicians, they use only two or three factors. Personality factors that are normally independent — such as energy and openness — were highly correlated in a more simplified view of personality.

Political candidates gain intense media exposure over an extended period of self-promotion designed to portray them as trustworthy experts with many admirable personality traits[5]. Such public exposure is intended to lead to clearly articulated perceptions rather than stereotypical evaluations by the electorate[6].

The nature of campaign information is unique as a basis for forming impressions of personality, as it is packaged by supporters and opponents as pros and cons (favourable or condemning) designed to simplify the ultimately dichotomous decision of how to vote. The selective mental processing and filtering by the electorate of the mass of discrepant input about political candidates must in the end justify each person's one vote: be it for or against. Therefore, we predicted that personality judgements about political candidates would likewise be constricted to involve a limited number of factors rather than the usual five.

We first studied the personality judgements of a sample of 2,088 Italian adults, of diverse ages, education and political views. Leading party politicians were evaluated by 1,257 respondents, and another 831 evaluated their own personalities and those of several celebrities. Judgements were made from a list of 25 adjectives that are markers of the five-factor model. Each adjective (for example, enterprising, reliable, truthful) was rated on how characteristic it was of each target on a seven-point scale, and those ratings were factor-analysed[7]. The analysis reduces the scores to a minimal number of correlated groups of traits within factors that are independent of each other.

Ratings were made of two Italian political candidates (Silvio Berlusconi and Roman Prodi), an international celebrity (skiing hero Alberto Tomba) and a famous Italian television personality (Pippo Baudo).

Table 1 reveals three clear results: (1) respondents' personality portraits of themselves require the five-factor solution, as found in earlier research; (2) personality judgements of national celebrities also require five factors; but (3) personality judgements of political candidates are drastically reduced to only two factors, despite many significant differences between their personalities.

The first of the two stable[8] personality factors for politicians has been named energy/innovation (which is a blend of energy and openness), and the second factor is honesty/trustworthiness (a blend of agreeableness, conscientiousness and stability).

These findings can be applied more generally, as shown by our replication study with 195 US college students. These students rated their own personalities after having rated Democratic President Bill Clinton and Republican presidential candidate Bob Dole, along with basketball star 'Magic' Johnson. The same 25 five-factor model marker adjectives were used as in our Italian study.

Table 1 shows that this different sample replicates the basic factor patterns found in the larger Italian sample: self-ratings and the ratings of the popular basketball player (among basketball fans) use all five factors, but judgements of the politicians are restricted to only three factors (among potential voters).

Finally, the percentage of total variance explained by each factor solution (2, 3 or 5) for each target personality, for both samples, is a high, nearly identical, average of 60 per cent.

We conclude that, by adopting a simplifying method of judging political candidates' personalities, voters use a cognitively efficient strategy for coding the mass of complex data, thus combating informational overload[9]. Doing so helps them to decide how to cast their vote.

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Table 1 **Factor composition for target personalities**

	Factors 1	2	3	4	5	Variance explained (%)
Italian sample						
Self (n=827)	E	O	A	C	S	56
Athlete (n= 829)	A	E	C	O	S	57
TV star (n=830)	C	E	S	O	A	60
Politicians						
Berlusconi (n=1,257)	A+C+S	E+O	–	–	–	61
Prodi (n=643)	A+C+S	E+O	–	–	–	64
US sample						
Self (n=195)	A	C	E	O	S	57
Athlete (n=81)	S	C	O	A	E	61
Politicians						
Clinton (n=127)	E+O	C+A	?	–	–	57
Dole (n=127)	C+A	?	E+O	–	–	62

Factors 1–5 are arranged in order of the amount of variance in ratings, with Factor 1 explaining the most variance and Factor 5 the least. E, energy; O, openness; A, agreeableness; C, conscientiousness; S, emotional stability; and '?', an uninterpretable factor. Adjectives used to describe the politicians' two factors, normally attributed to the factors shown in parentheses, are – **energy/innovation:** enterprising (E), active (E), self-assured (E), energetic (E), cheerful (E), innovative (O), creative (O), inventive (O), smart (O), modern (O), efficient (C), optimistic (S), confident (S), cordial (A); and **honesty/trustworthiness:** sincere (A), truthful (A), loyal (A), responsible (C), reliable (C), precise (C), persistent (C), poised (S), peaceful (S), stable (S), generous (A).

1. Costa, P. T. & McCrae, R. R. *The NEO Personality Inventory Manual* (PAR, Odessa, 1985).
2. Briggs, S. *J. Personality* **60**, 254–293 (1992).
3. Caprara, G. V., Barbaranelli, C., Borgogni, L. & Perugini, M. *Personality Indiv. Diff.* **15**, 281–288 (1993).
4. Goldberg, L. R. *Am. Psychol.* **48**, 26–34 (1993).
5. Simonton, D. K. *Why Presidents Succeed: A Political Psychology of Leadership* (Yale Univ. Press, New Haven, 1987).
6. Pierce, P. *Political Psych.* **14**, 21–35 (1993).
7. Cattell, R. B. & Vogelmann, S. *Multivariate Behav. Res.* **12**, 289–325 (1977).
8. Tucker, L. R. *A Method for Synthesis of Factor Analysis Studies* (Dept of the Army, Washington DC, 1951).
9. Fiske, S. & Taylor, S. *Social Cognition* (McGraw Hill, New York, 1991).

*More detailed methods and additional results are available from G. V. C. at caprara@axrma.uniromal.it

Introduction to Homotopy Theory

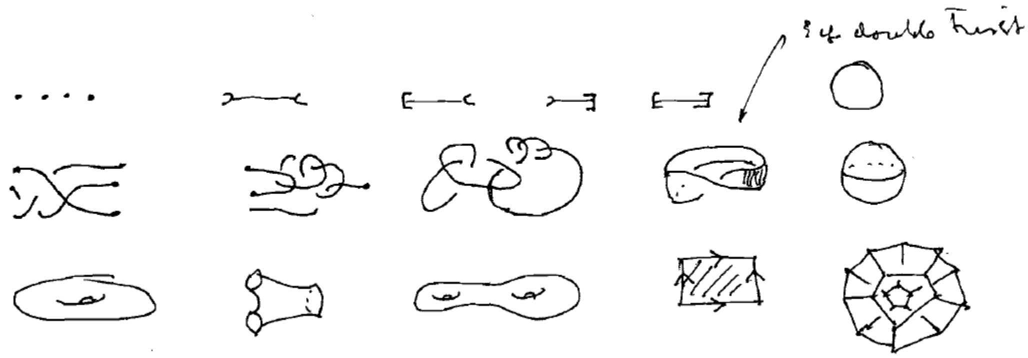
Topology is a very intuitive subject. But: just because you can formalize intuitive notions doesn't mean that's the best way into the subject.

(Ex. The **(blank)** numbers Dedekind Cuts)

If any point you think a formal definition would be helpful, don't hesitate to ask.

What does it mean to say that two different things are the same? Algebraic Topology started as a kind of geometry, but has morphed into the study of different kinds of equivalence.

Some Spaces



Some keywords: boundary, compact, connected, dimension, embedding, manifold, orientable, product space, quotient space, subspace.

Some important symbols: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

[Group Theory, Algebra]

Descartes: Spaces vs. Functions defined on them.

$\mathbb{R}$ – spaces and algebras
X nice (eg compact) space, $C(X, \mathbb{R})$ = continuous real valued functions on $X \ni f : X \to \mathbb{R}$ (a Banach Algebra $\in$ Analysis)

Ex. $\mathbb{R}[x, y] \to C(\mathbb{R}^2, \mathbb{R})$, 2-variable polynomials to Continuous functions on Real plane.

Doesn't simplify our lives enough! It conveys too much information, does not help with question of what it means to be "the same".]
Personal names: eg, persons, animals, hurricane,fires and earthquakes.

Fundamental Group of a Space.

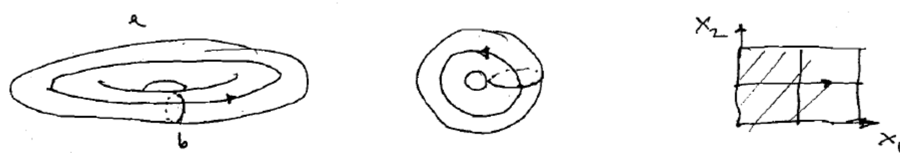
This is going to be a course on homotopy theory, but it is going to center around homology. Homotopy groups are more fundamental than homology groups; they're easier to define, and contain more information. But that just makes them all that much more intractable.

Homology Groups

There are easier to work with than to define, and in many ways are most accessible when approached axiomatically.

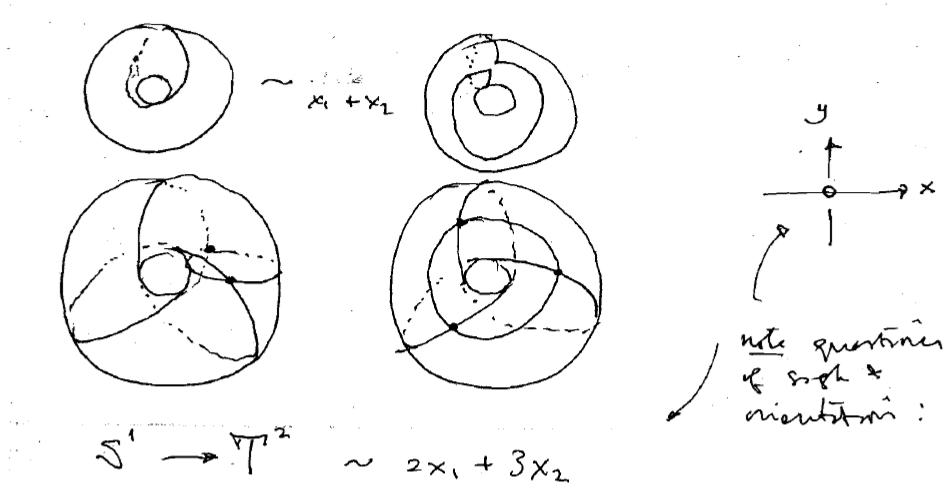
Ex. The two- dimensional torus

There are basically two kinds of closed path on a torus:



Any closed path on a torus can be written as a "sum" of these basic paths: (Astrids section 1.24, pg 47)
$S' \to S' \times S'$
$z \mapsto (z^r, z^p)$

Homotopy Theory



(Note the Mazda symbols is a trefoil on a torus)

Claim: For any (reasonable) space X, there exists (f. gen) abelian groups $H_i(X, \mathbb{Z})$, $i = 0, 1, 2, \ldots$

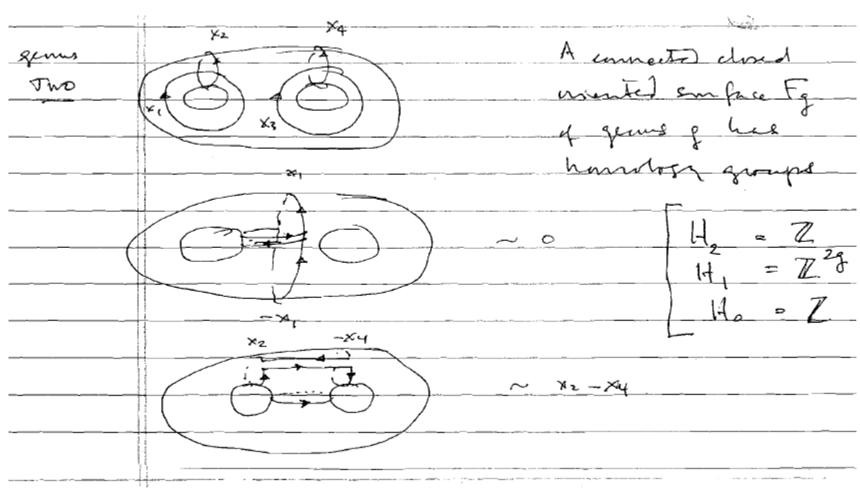
whose elements are equivalent classes of individual geometric objects in X with various properties, to be spelled out as i-dimensional cycles.

Ex $H_0(T^2) = \mathbb{Z}$

$H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$

$H_2(T^2) = \mathbb{Z}$

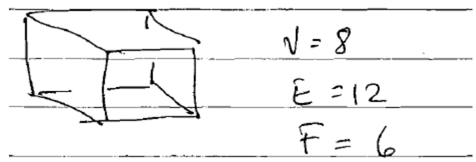
$H_i(X, \mathbb{Z}) = 0$, $i >$ dim X where a connected, closed, oriented surface with "holes" = surface of genus g]

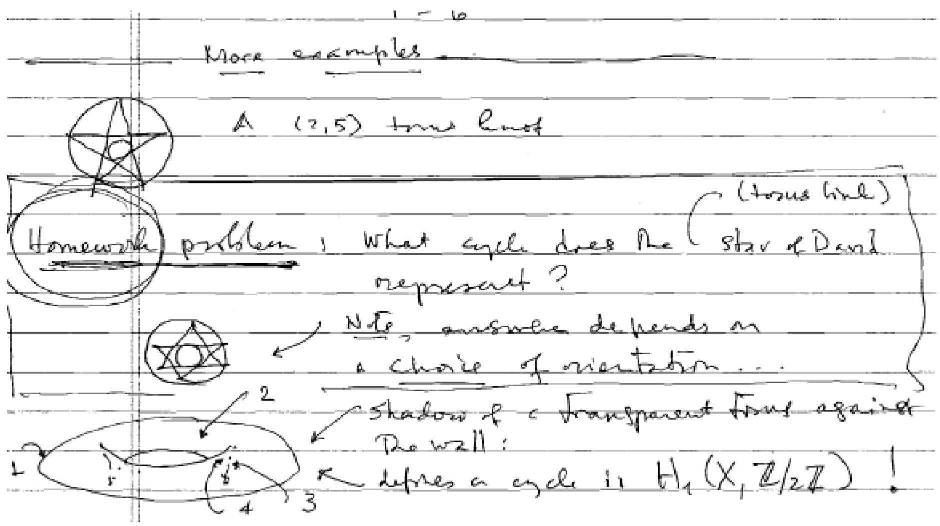


The *Euler characteristic* of a topological space X equals $\sum(-1)^k \text{rank } H_k(X, \mathbb{Z}) := \chi(X)$

Example: $\chi F_g = 1 - 2g + 1 = 2(1 - g)$.

Euler showed that any polyhedral decomposition of the 2- sphere, with V vertices, E edges, and F faces has $V - F + E = \chi(X) = 2$.)

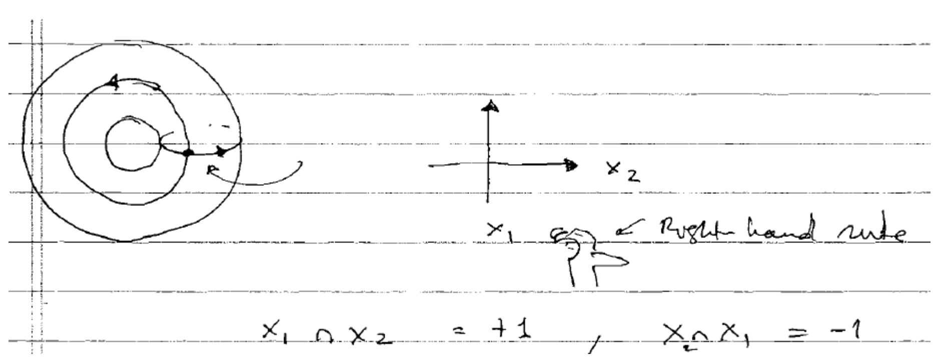




Intersection Theory

In $n-$ dimensional Euclidean Space $\mathbb{R}^n$, a set of $p-$ *generic* equations define a subspace of *codimension* p, i.e, dimension $n-p$.
Ex $\{x_1 = 0, \ldots, x_p = 0\}$
and a subspace of codimension p and a subspace of codimension q intersect (generically) in a subspace codimension $p+q$! (This is more or less Bezout's Theorem in Algebraic Geometry.)

In an oriented closed $n-$ manifold, there is an *intersection product*:
$H_{n-p}(M) \otimes H_{n-q}(M) \to H_{n-(p+q)}(M)$



$$H_1(T^2) \otimes H_1(T^2) \to \mathbb{Z} = H_0(T^2) \begin{cases} x_1 \cap x_2 & = +1 \\ x_1 \cap x_1 & = 0 \\ x_2 \cap x_2 & = 0 \end{cases}$$

More generally, $H_1(F_g), \mathbb{Z}) = \mathbb{Z}^{2g})$, with a basis: $x_1, x_2, \dots x_{2g-1}, x_{2g}$

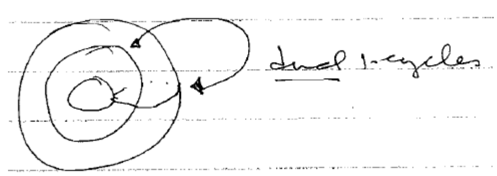
Intersection Product:
$x_{2i-1} \cap x_{2i} = 1$ for $i = 1, \dots, g$
$x_i \cap x_k = 0$ if $|i-k| \neq 1$

$H_i(F) \otimes H_k(F) \to H_{2-(i+k)}(F)$:

$\mathbb{Z} = H_0(F) \otimes H_k(F) \to H_{2-k}(F)$

Poincare Duality Theorem: This is an isomorphism (There is some nontrivial fine print!.)

Ex. $x_{2i-1} \leftrightarrow x_{2i}$ in a surface:



Ex. 1 point = generator of $H_0(F)$ is dual to the whole manifold, $[F] \in H_2(F)$

Summary:

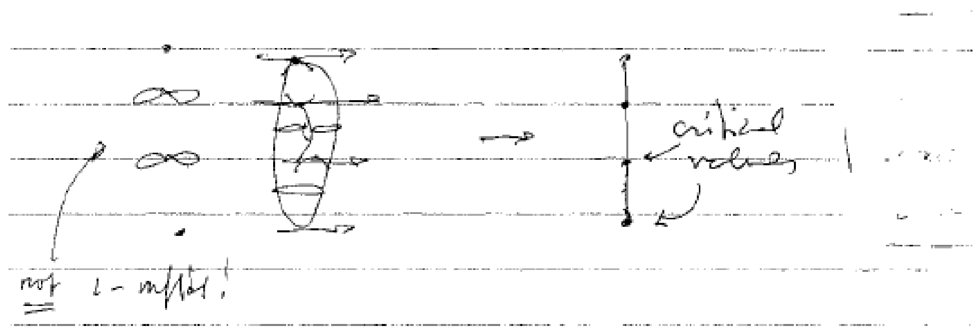
A general space has a collection $H_i(X, \mathbb{Z})$ of abelian homology groups.

When X is an (orientable) manifold, this collection has an extra product structure, making it into a kind of commutative algebra.

Important question to bring into focus: How do these invariants vary, as X changes.

The answer requires a digression into Category Theory.

Theorem (Sard's Theorem)**.** *Let $f : \mathbb{R} \to \mathbb{R}^m$ by C^k (that is, k times continuously differentiable), where $k \geq max\{n - m + 1, 1\}$. Let C denote the critical set of f, which is the set of points $x \in \mathbb{R}^n$ where the Jacobian matrix of f has rank ¡ m. Then the image of $f(X)$ has* *measure 0.*



Implicit for theorem: if $v \in \mathbb{R}^p$ is not critical, then $f(x)$ is a higher-dimensional manifold. Proof: v not critical implies that for all $x \in overlinef'(x)$, rank $f'(x) = p$ is maximal. There exists a neighborhood V of dimension mp if $x \in \overline{f}(v)$.

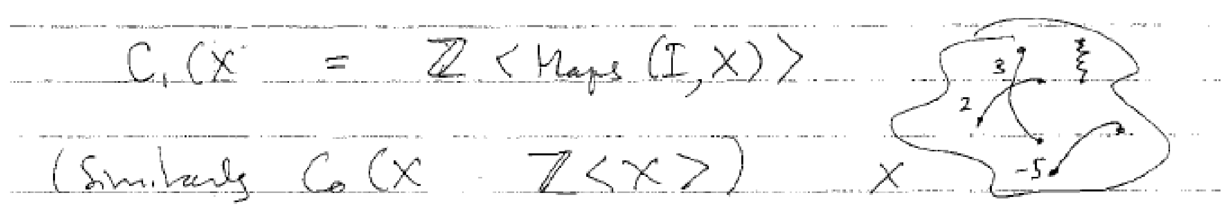
Corollary 0.1. *$f : M^m \to N^n$ smooth, then almost every $v \in N$, $\overline{f}'(v)$ is a manifold.*

(Open question: infinite dimensional case?)

Homology in (very) low dimensions.
X a set,

$$\begin{aligned} \mathbb{Z}\langle X\rangle &= \text{free abelian group generated by } X \\ &= \bigoplus_{x\in X} \mathbb{Z}\langle x\rangle \\ &\ni \sum_{\text{finite}} n_i < x > \text{ with } n_i \in \mathbb{Z} \end{aligned}$$

X a space, Maps(I,X) $= \{s : I \to X \text{ continuous}\}$, where $I = [0, 1]$ = unit interval.



Definition.

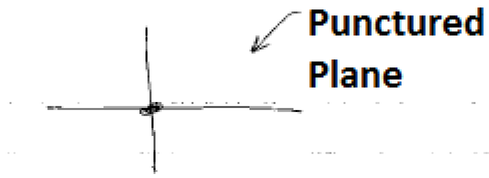
$$\partial_1 : C_1(X, \mathbb{Z}) \to C_0(X, \mathbb{Z})$$

is the boundary homomorphism, where

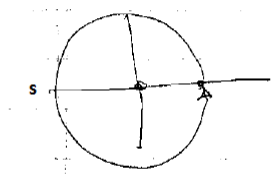
$$\partial_1 \left(\sum n_i\langle s_i\rangle\right) = \sum n_i[\langle s_i(1)\rangle - \langle s_i\langle 0\rangle]$$

(German Zyklus):

$$\begin{aligned} Z_1(X,\mathbb{Z}) &= \text{ group of 1-cycles} \\ &= \ker\, \partial_1 \\ &= \{z \in C_1 \,|\, \partial_1(z) = 0\} \end{aligned}$$

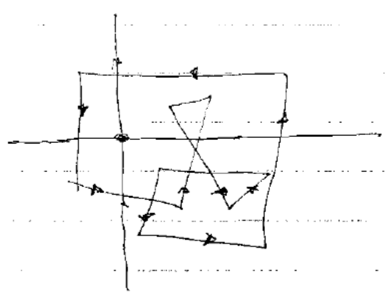


Ex. $X = \mathbb{R}^2 - 0$

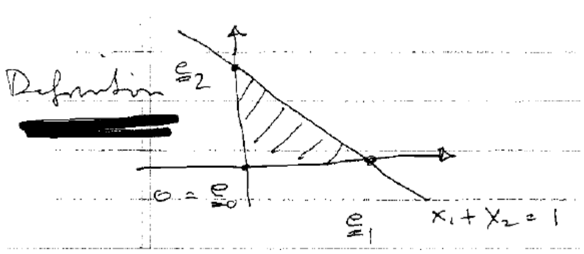


$s(t) = (\cos 2at, \sin 2at) : I \to \mathbb{R}^2 - 0 = X \in Z_1(\mathbb{R}^2 - 0, \mathbb{Z})$
(because $\partial s = \langle(0,1)\rangle - \langle(0,1)\rangle = 0$).

Another example:

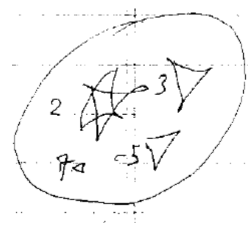


$n-$ simplex
($n \geq 2\, :\, \Delta^1 = I, \Delta^0 = \text{pt}$)



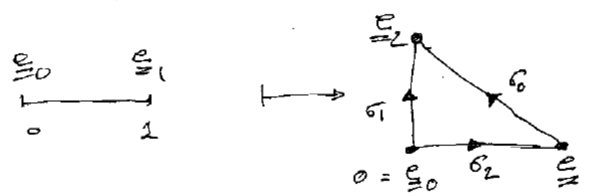
$\triangle^n = \{x \in \mathbb{R}^n | x_1 \geq 0, \sum x_i = 1\} = \text{standard}$

Note: The $n-$ simplex has $n+1$ vertices!
$C_n(X,\mathbb{Z}) = \mathbb{Z}\langle(Maps(\triangle^n, X)\rangle$ = group of n-chains in$X$



(More generally: convex hull of n linearly independent vectors $= n-$ simplex)

Define



by

$\sigma_1(t) = (1-t)\mathbf{e}_1 + t\mathbf{e}_2$
$\sigma_2(t) = (1-t)\cdot 0 + t\mathbf{e}_2$
$\sigma_3(t) = (1-t)\cdot 0 + t\mathbf{e}_2$

Since linear maps between simplexes are determined where they send vertices, these maps can be defined by

$\sigma_0 : \{0,1\} \to \{1,2\}$
$\sigma_1 : \{0,1\} \to \{0,2\}$
$\sigma_2 : \{0,1\} \to \{0,1\}$, or more generally,

$\sigma_i = \{0_1 = \ldots = n\} = \{0,\ldots,i-1\ldots,\hat{i},i+1,\ldots,n\}$

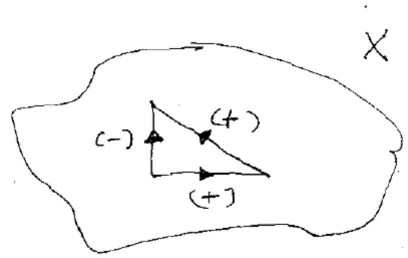
The notation meaning that i is omitted from the sequence on the right: for example; σ_0 sends $\{0,1\} \neq \{1,2\}$. **(check)**

<u>Definition:</u> $\partial_n : C_n(X,\mathbb{Z}) \to C_n(X,\mathbb{Z})$ sends $S : \triangle^n \to X$ to

$$\partial_n(S) = \sum_{i=0}^{i=n} (-1)^i S \circ \sigma_i$$

For ex., if $S : \triangle^2 \to X$ then $\partial_2 S$ is the 1-chain

$S \circ \sigma_2 + S \circ \sigma_0 - S \circ \sigma_1$:



Note that:

$$\begin{aligned} \partial\sigma_0 &= \langle 2\rangle - \langle 1\rangle \\ -\partial\sigma_1 &= -\langle 2\rangle + \langle 0\rangle \\ \partial\sigma_2 &= \langle 1\rangle - \langle 0\rangle \end{aligned}$$

So $\partial_1 \cdot \partial_2 = 0$:

See lemma 2.1 of §2.1 (p 105) of Hatcher, §9.1.2 (p 224) of tom Dieck

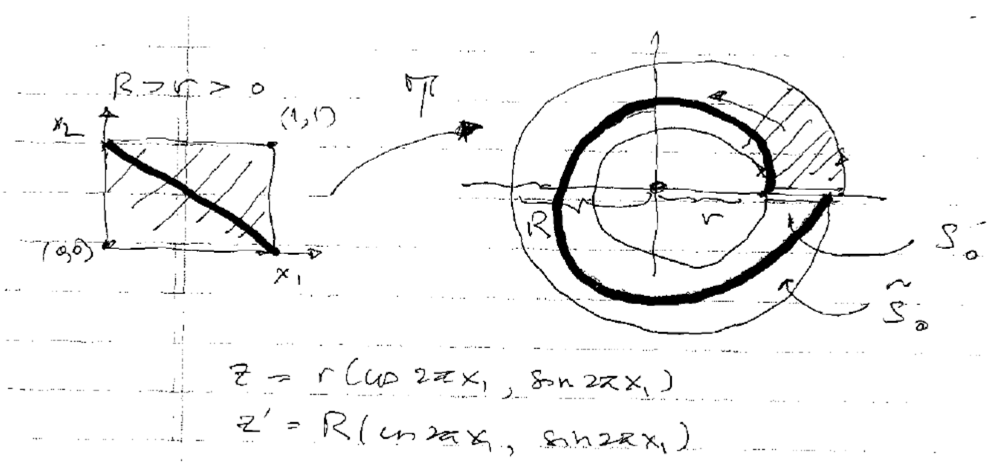
$\therefore \partial_2 S \in \ker \partial_1, \forall S$, so image $\partial_2 \subset \ker \partial_1 = B_1(X,\mathbb{Z})$ = group of boundary cycles.

Definition: $H_1(X,\mathbb{Z}) := Z_1(X,\mathbb{Z})/B_1(X,\mathbb{Z})$ = quotient abelian group.

More generally, $H_n(X,\mathbb{Z}) = Z_n/B_n = \ker \partial_n / \text{im } \partial_{n+1}$

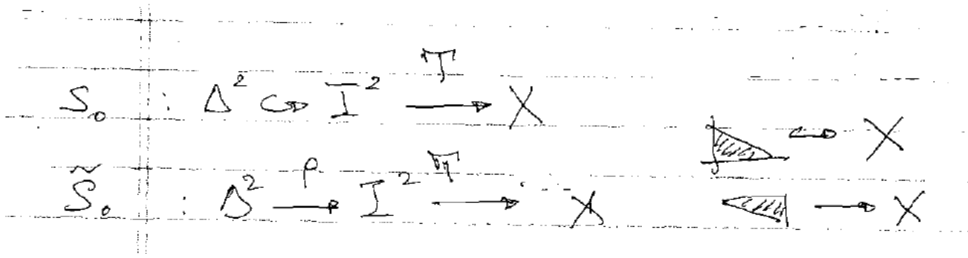
Alt: Cycles z, z' are homologous, $(z \sim z') \Leftrightarrow \exists$ 2− chain S such that $\partial_2 S = z - z'$.

Ex. $T : I^2 \to \mathbb{R}^2 - 0$ where $T(x_1,x_2) = (x_2 R + (1-x_2)r)(\cos 2\pi x_1 \sin 2\pi x_1)$



Let $\rho : I^2 \to I^2$,
$\rho(x_1,x_2) = (1-x_2, 1-x_1)$ = reflection across $x_1 + x_2 = 1$
$\rho \circ \rho$ = identity, thus $x_1 + x_2 = 1$, fixed.

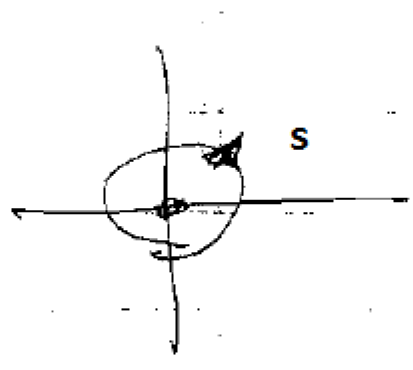
Define



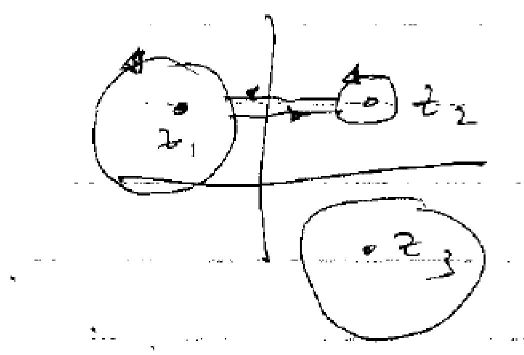
Claim: $S_0 + \widetilde{S}_0$ [blank]

$\boxed{\partial S = z - z'}$ or $S : z \sim z'$

This suggests (though does not prove) that $H_1(\mathbb{R}^2 - 0, \mathbb{Z}) = \mathbb{Z}$ is generated by the class of the cycles.



More generally, it suggest that $H_1(\mathbb{R}^2 - \{z_1, \ldots, z_n\}, \mathbb{Z}) \cong \mathbb{Z}^n$. (plane-$\{n\,$distinct points$\}$)



This goes back to Cauchy's Theorem, in the theory of functions of one complex variable.