Everything you always wanted to know about trig*

Explained by
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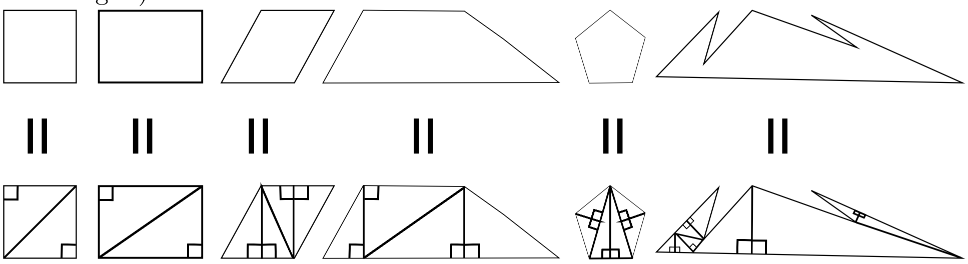
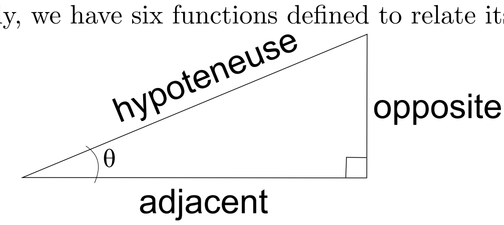
*BUT WERE AFRAID TO ASK.

Any figure drawn on a piece of paper with straight edges can be split up into right triangles. Because of this, studying squares, rectangles, rhombi, trapezoids, and all other regular and irregular polygons, can be seen as studying right triangles only. This is why there is a subject in school called *trigonometry* (measurement of triangles).



To study a right triangle fully, we have six functions defined to relate its sides and angles.



$$\sin\theta = \frac{\text{opposite}}{\text{hypoteneuse}} \qquad \csc\theta = \frac{\text{hypoteneuse}}{\text{opposite}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypoteneuse}} \qquad \sec\theta = \frac{\text{hypoteneuse}}{\text{adjacent}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Really, $\sin\theta$ and $\cos\theta$ contain all of the information, the others are just shorthand.

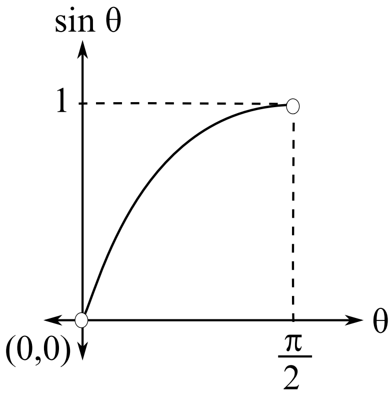
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

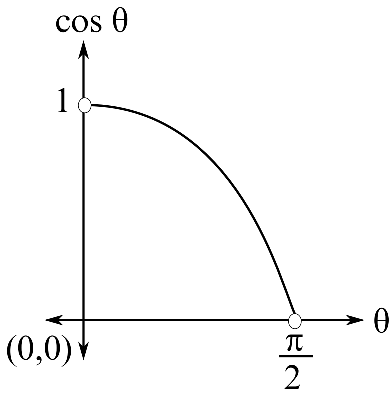
You can memorize which is which by the old stand-by SOH-CAH-TOA*. i.e.

$$\underline{\text{S}}\text{ine} = \underline{\text{O}}\text{pp}/\underline{\text{H}}\text{yp} \quad | \quad \underline{\text{C}}\text{osine} = \underline{\text{A}}\text{dj}/\underline{\text{H}}\text{yp} \quad | \quad \underline{\text{T}}\text{angent} = \underline{\text{O}}\text{pp}/\underline{\text{A}}\text{dj}$$

The fact that $s = \frac{1}{c}$ and $c = \frac{1}{s}$, tan, and cot, you'll just have to get used to.

Since these were defined with θ being a non-right angle in a right triangle, they only make sense when $0 < \theta < \frac{\pi}{2}$. The only thing you ever do with degrees is translate them into radians: ($\pi = 180° \Rightarrow 30° = \frac{180°}{6} = \frac{\pi}{6}$ etc.). So, graphs of sin and cos would look like



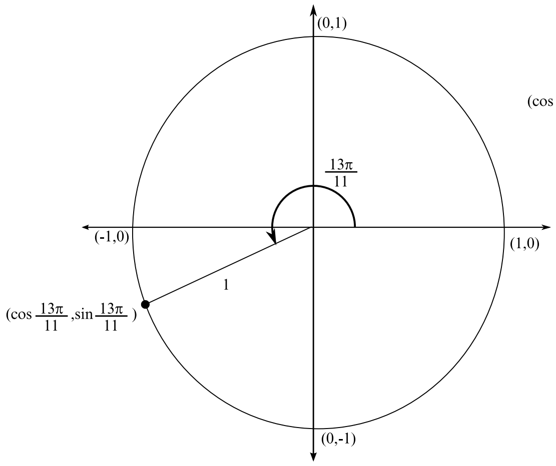


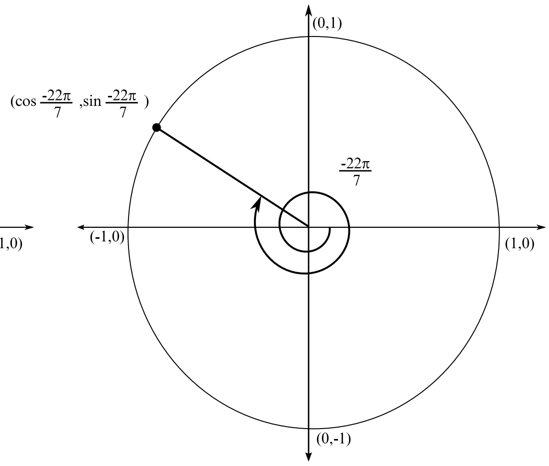
* Or you can use "$\underline{\text{S}}$illy $\underline{\text{O}}$ld $\underline{\text{H}}$arry $\underline{\text{C}}$arried $\underline{\text{A}}$ $\underline{\text{H}}$orse $\underline{\text{T}}$o $\underline{\text{O}}$ur $\underline{\text{A}}$partment" if it bothers you that Sohcahtoa is meaningless aside from its similarity to Krakatoa…

Even though triangles with $\theta = 0$ or $\theta = \frac{\pi}{2}$ don't make sense, by the graphs, we can tell what the values should be. Here's an easy way to memorize all of the "important" values for sin, cos, and tan. Notice the pattern in

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	$\sqrt{0/4}$	$\sqrt{1/4}$	$\sqrt{2/4}$	$\sqrt{3/4}$	$\sqrt{4/4}$
$\cos\theta$	$\sqrt{4/4}$	$\sqrt{3/4}$	$\sqrt{2/4}$	$\sqrt{1/4}$	$\sqrt{0/4}$
$\tan\theta$	$\sqrt{0/4}$	$\sqrt{1/3}$	$\sqrt{2/2}$	$\sqrt{3/1}$	$\sqrt{4/0}$

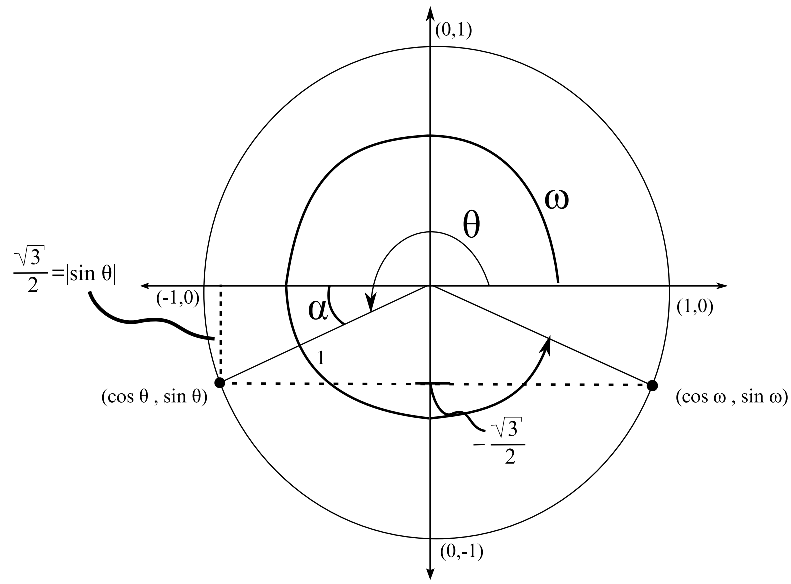
So
$\cos\frac{\pi}{4} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
To generalize these definitions so that θ can be any number, define $\sin\theta$ and $\cos\theta$ with a circle





Given an angle θ, find the point corresponding in the <u>unit circle</u> — the x-coordinate will be $\cos\theta$ and the y-coordinate will be $\sin\theta$. This is very nice for solving problems like

$$\sin\theta = \frac{-\sqrt{3}}{2}$$



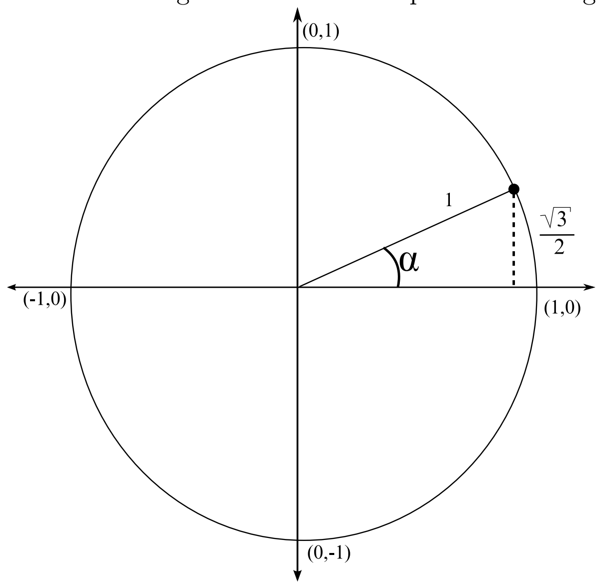
All solutions to the problem are

$$\begin{array}{cccc} \theta, & \theta+2\pi, & \theta+4\pi, & \dots \\ & \theta-2\pi, & \theta-4\pi, & \dots \\ \omega, & \omega+2\pi, & \omega+4\pi, & \dots \\ & \omega-2\pi, & \omega-4\pi, & \dots \end{array}$$

which is always shortened to

$$\theta + 2\pi k \text{ and } \omega + 2\pi k, \qquad k \in^{*} \mathbb{Z}^{\dagger}$$

In our case, we can examine the same triangle moved to the I quadrant and figure out what α is.



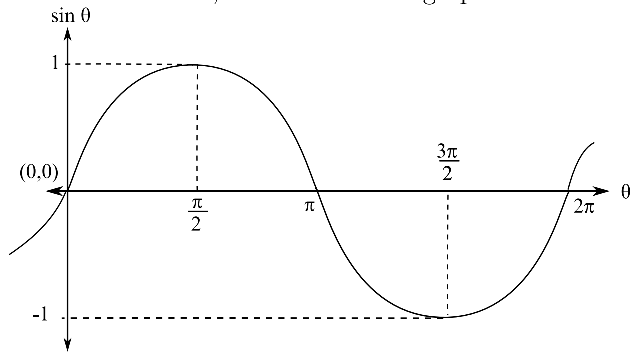
Looking at the table, see that α is $\frac{\pi}{3}$, so $\theta = \pi + \alpha = \pi + \frac{\pi}{3} = \frac{4}{3}\pi$ and $\omega = 2\pi - \alpha = 2\pi - \frac{\pi}{3} = \frac{5}{3}\pi$. So all solutions would be these two angles $+2\pi k, \quad k \in \mathbb{Z}$. A calculator, when asked for $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ only tells you one of these angles, namely $-\frac{\pi}{3}$.

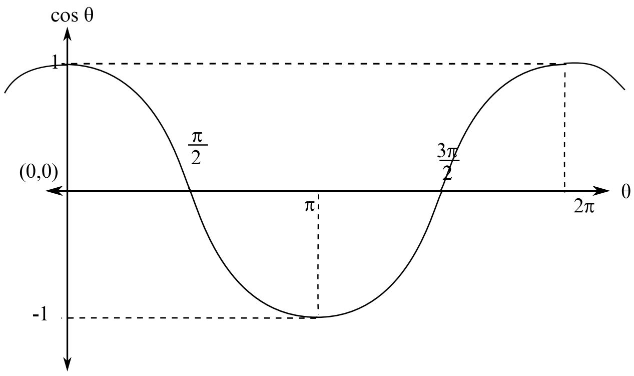
* $\in$ means "member of"

† $\mathbb{Z}$ means "the integers", i.e. $\dots, -2, -1, 0, 1, 2, \dots$

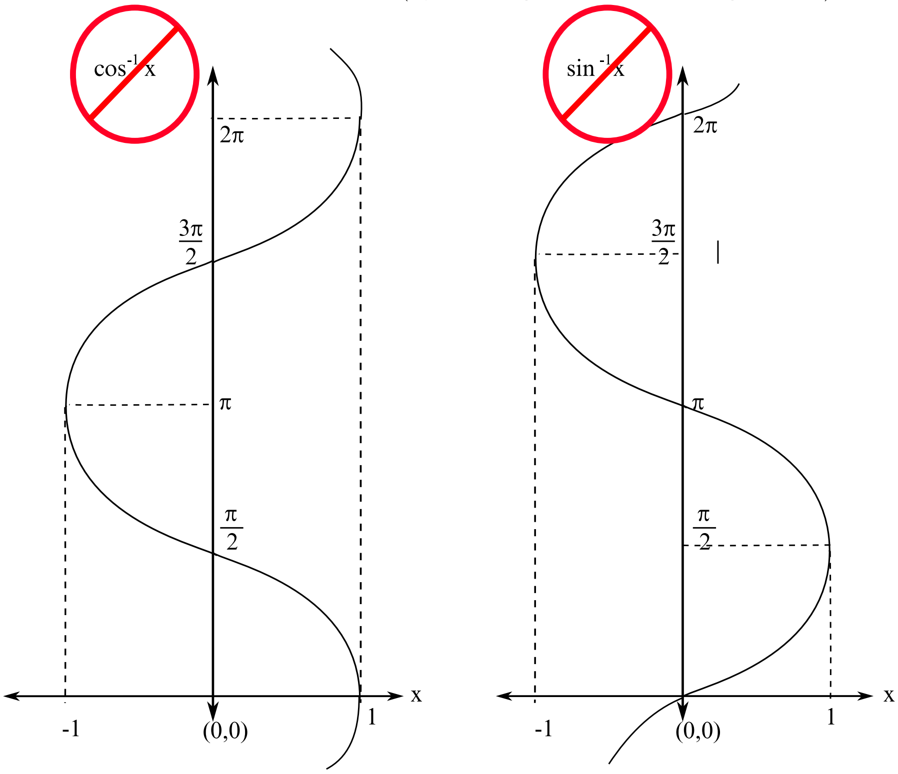
So "$k \in \mathbb{Z}$" means "k in the set of integers" i.e. "$k = \dots, -2, -1, 0, 1, 2, \dots$"

With the circular definition of sin and cos, we can have the graphs we're more accustomed to



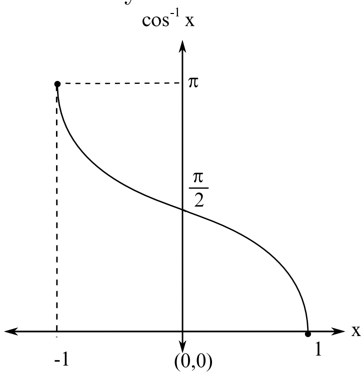


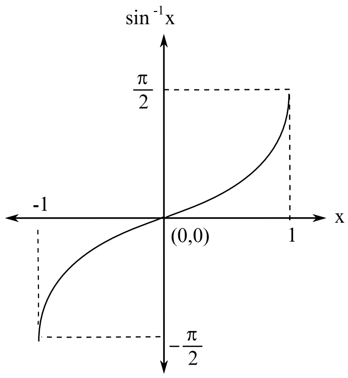
Notice that $\sin^{-1}$ and $\cos^{-1}$ would have to be (by flipping about the 45-degree line)



but these wouldn't be functions (since they'd fail the vertical line test), so a decision had to be made.

This is why $\sin^{-1}$ and $\cos^{-1}$ look like





These are the functions your calculator uses, which explains why it misses ∞ of the answers to a simple question like:

Given

$$\sin\theta = \frac{-\sqrt{3}}{2},$$

what is θ?

Now, how do we memorize all of the trigonometric identities we're expected to know? First memorize the "pythagorean identity"

$$\sin^2\theta + \cos^2\theta = 1$$

Immediately, you can see that this means

$$\sin^2\theta = 1 - \cos^2\theta$$
$$\cos^2\theta = 1 - \sin^2\theta$$

Derive the two similar identities by dividing by $\sin^2\theta$ on both sides of the pythagorean identity

$$1 + \cot^2\theta = \csc^2\theta$$

and dividing by $\cos^2\theta$ on both sides of the pythagorean identity

$$\tan^2\theta + 1 = \sec^2\theta$$

Now memorize the angle addition formulae and you don't have to memorize anything else. The rest can be derived. These are easier to understand when you say them out loud than when you read them on paper. So, say out loud "Sine sucks"

$$\sin(x+y) = \sin x\cos y + \cos x \sin y \qquad (\text{s} \quad \text{c} \quad \text{c} \quad \text{s} \quad = \quad \underline{\text{succs}} \quad = \quad \text{sucks})$$

"Cosine kicks"

$$\cos(x+y) = \cos x\cos y - \sin x \sin y \qquad (\text{c} \quad \text{c} \quad \text{s} \quad \text{s} \quad = \quad \underline{\text{cicss}} \quad = \quad \text{kicks})$$

make sure to memorize which one has a minus sign in the middle and which doesn't.

Now you can derive every other identity anybody has ever expected of you

$$\underline{\sin(2x)} = \sin(x+x) = \sin x\cos x + \cos x\sin x = \underline{2\sin x\cos x}$$

$$\begin{aligned}
\underline{\cos(2x)} &= \cos(x+x) = \cos x\cos x - \sin x\sin x = \underline{\cos^2 x - \sin^2 x}\\
&= \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x) = \cos^2 x - 1 + \cos^2 x = \underline{2\cos^2 x - 1}\\
&= (1-\sin^2 x) - \sin^2 x = \underline{1 - 2\sin^2 x}
\end{aligned}$$

From the last two, you can also see that

$$\begin{aligned}
\cos 2x = 1 - 2\sin^2 x &\Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}\\
= 2\cos^2 x - 1 &\Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}
\end{aligned}$$

Work all of the above derivations out on a piece of paper yourself. Once you've done it once, you can't help but be able to do them in the future.