Basic Algebraic Geometry II: Schemes and Complex Manifolds
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SOLUTIONS

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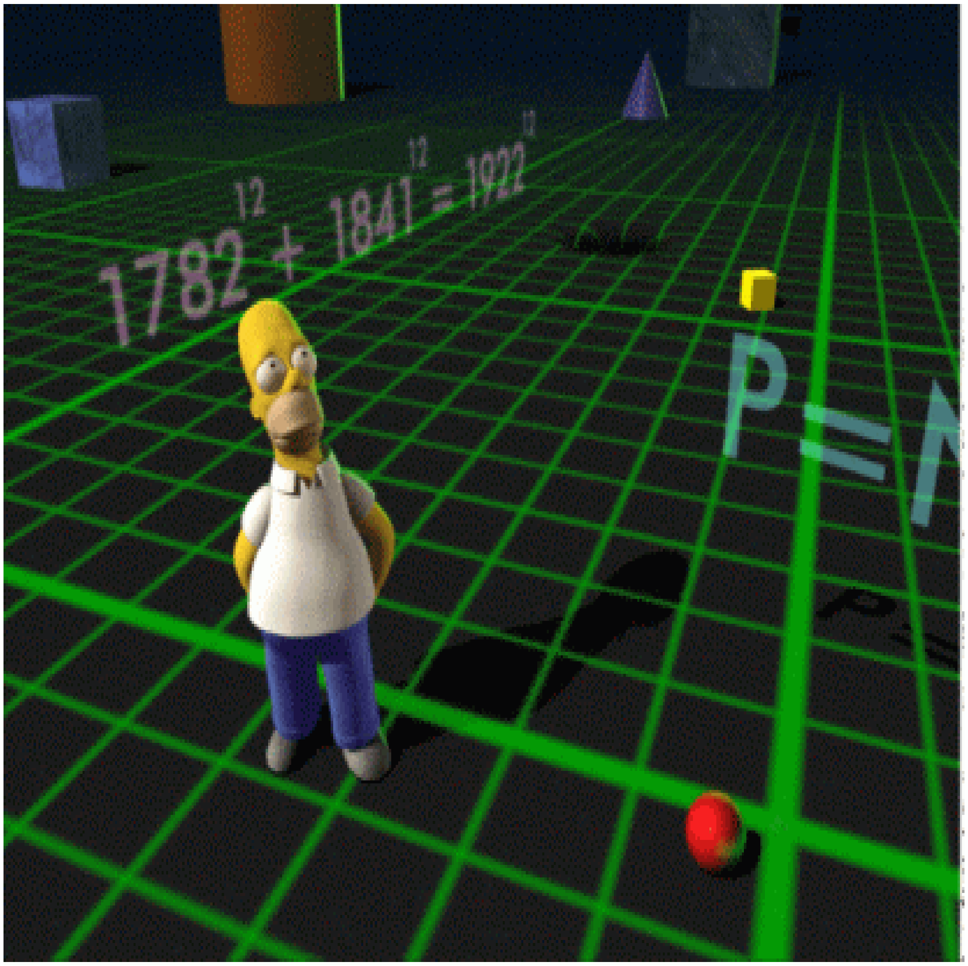

1 Chapter V: Schemes

1.1 The Spec of a Ring

1. Since N is an ideal of A, contained in every prime ideal, then the argument on page 10 applies showing that ${}^a\varphi$ is a homeomorphism of $\text{Spec}(A/N)$ to the closed set $V(N) = \text{Spec } A$.

2. See Corr 1.4 in Hartshorne which says that an affine variety is irreducible iff its ideal is a prime ideal. Here Spec A is $V(0)$ which would be irreducible iff (0) is prime, ie no zerodivisors.

3. Let $p \in \text{Spec } A$. Since B is integral over A, there exists a q in Spec B such that $q \cap A = p$ (AM Thm 5.10).

4. No since maximal ideals do not contract to maximal ideals in general. Ex: $\varphi : \mathbb{Z} \hookrightarrow \mathbb{Q}, (0)^c = 0$. However, if B is integral over A, then yes, the contraction of maximal ideals is again maximal by (AM Cor 5.8). This corresponds to the notion that that finite maps take closed sets to closed sets.

5. $p \in \overline{\varphi^a(V(E))}$ iff there exists $p' \in^a \varphi(V(E))$ such that $p' \subseteq p$. Therefore $q \supseteq E$ such that ${}^a\varphi(q) = p'$, so $p' \supseteq \varphi^{-1}(E)$. So $p' \in V(\varphi^{-1}(E))$. This also $p \in V(\varphi^{-1}(E))$. So $\overline{{}^a\varphi(V(E))} \subseteq V(\varphi^{-1}(E))$. The other inclusion is false... Take $\varphi : \mathbb{Z} \hookrightarrow \mathbb{Q}$ and $E = (2)$. Then the left hand side is empty and the right hand side is $\{2\}$.

6,7 The u_i's in this problem are called orthogonal idempotents. Claim: The following are equivalent: 1) Spec A is disconnected, 2) there exists non-zero orthogonal idempotents u_1, u_2 in A, 3) $A \cong A_1 \times A_2$. Proof: $1 \Rightarrow 3$: Let $X = \text{Spec } A = X_1 \cup X_2$, where $X_1 = V(r(a))$ and $X_2 = V(r(b)) = V(r(a) \cap r(b))$. Therefore $r(a)$ and $r(b)$ are comaximal, so $r(a) \cap r(b) \in N$, where r is the radical of an ideal, and N the nilradical, a,b ideals in A. Fix $a \in a, b \in b$ such that $a + b = 1$. Then $a, b \in N$ since $X_a \subseteq X_1$, $X_b \subseteq X_2, X_a \cap X_b = \emptyset$. If $a^n b^n = 0, a^n, b^n \neq 0$ since $(a + b)^n = 1, (a^n + b^n = 1 - abf$ for $f \in A$. $abf \in N$ and since $N \subseteq J$ (Jacobson Radical)$, 1 - abf \in A^*$. Therefore for some $g \in A^*, a^n g + b^n g = 1, a^n g b^n g = 0$, and since clearly $a^n g, b^n g \neq 0, 1$, then $a^n g(1 - a^g) = 0, a^n g \neq 0, 1$. Therefore they are idempotent and so is $b^n g$. let $A_1 = A/(a), A_2 = A/(b)$. Then $A \cong A_1 \times A_2$.

$(3 \to 2)$: $(0, 1)^2 = (0, 1), (1, 0)^2 = (1, 0)$

$(2 \to 1)$: Let e be idempotent. $e(1 - e) = 0$, and $X_e \cap X_{1-e} = \emptyset$, so $X_e \cup X_{1-e} = X$

8. Let $A = k[x_1, \ldots x_n]/a$. Then $f \in A$ is nilpotent iff f vanishes on $Z(a)$ by the Nullstellensatz iff $f \in m_x \forall x \in Z(a)$. But $\{m_x | \, x \in Z(a)\}$ are precisely the maximal ideals of A.

9. Let $B = \mathbb{Z}[T]$. $B_p/pB_p \cong A_p/pA_p \cong \mathbb{Z}_p$. $pB_p/(pB_p)^2 = (p,T)/(p^2,pT,T^2) \cong \mathbb{Z}_p \times Z_p$ as a $\mathbb{Z}_p$-vector space so has dimension 2. There is a surjection $pB_p/(pB_p)^2 \twoheadrightarrow pA_p/(pA_p)^2$ sending $\frac{f}{g} \mapsto \frac{f}{g}$. This is a $\mathbb{Z}_p$-linear map so dim $pA_p/(pA_p)^2$ is 1 or 2. The dimension of A_p is 1 since p has height 1 in A by the Hauptidealsatz (AM 1 p 122), so p is a singular point iff dim $pA_p/(pA_p)^2 = 1$ iff the map is an injection. Note that the map takes F to 0 since $F = 0$ in A. If $p^2 \nmid F(0)$ or $p \nmid F'(0)$, then $F \notin (pB_p)^2$, so the map is not injective. If $p^2 | F(0)$ and $p | F'(0)$, then $F \in (pB_p)^2$. In fact, $F \in p^2$, where p is viewed as an ideal of B. Since were are taking F to 0 when modding out by $(pB_p)^2$, it does not matter that we initially modded out by f when defining A. So the map above is an isomorphism.

10. We can assume that the closed subset X of codimension 1 is irreducible. Let $X = V(E)$ and let $G = \prod G_i$ be in E with G_i irreducible. Then $V(G) \supseteq X$. And $V(G) = \bigcup V(G_i)$, so $X = \bigcup (V(G_i) \cap X)$. Since X is irreducible, $X \subseteq V(G_i)$ for some i. Set $F = G_i$. Then (F) is prime so $V(F)$ is irreducible and contains X. By the Hauptidealsatz, (F) has height 1, so dim $V(F) =$ dim X. Thus $X = V(F)$.

11. This is Prop 3.1 pg 37 in AM

1.2 Sheaves

1. Clearly a presheaf. Also, clearly a sheaf if $|X|$ or $|M|$ is a finite set. If both sets were infinite, define functions $u_i(x_i) = i$ for x_i in X. Then distinct local sections u_i and u_j are compatible since their supports don't intersect, but there is no global u defined on the infinite set $\{x_1, x_2 \ldots\}$ such that the restriction of u to $\{x_i\}$ is u_i since the image of such a section would be infinite. So this is not a sheaf.

2. Clearly a presheaf. Both sheaf conditions are satisfied since differential r-forms are defined locally and are compatible on intersections.

3. This is done in Prop 2.2b pg 71 in Hartshorne.

4. The restriction maps in the sheaf of locally constant functions take constant functions to constant functions so they factor through the subgroups of constant functions. So we get a well-defined restriction maps $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ if $U \supseteq V$. So $\mathcal{F}$ is clearly a presheaf. Now to determine the local ring $\mathcal{O}_x$: Every function is constant in a neighborhood of x, so all functions are constant in $\mathcal{O}_x$. Since we are modding out by the constant functions, $\mathcal{O}_x = 1$. Thus the sheafification is $\mathcal{F}' = 0$, the zero sheaf, where $\mathcal{F}'(U) = 0 \, \forall U$.

5. See Hartshorne p 110-111 for Definition and Prop 5.1

6. The tree of prime ideals for A is m as the root with single branches of prime ideals p_i coming off of it. If $\xi \in$ Spec A is a generic point, then the

tree must be $m - \xi$. So Spec $A = \{\xi, m\}$. $\{\xi\}^c = \{m\}$, which is closed, so ξ is an open point. The two open sets are $\{\xi\}$ and Spec A. Since $\{\xi\} \subset$ Spec $A, \mathcal{O}_\xi = \mathcal{O}(\xi) = A_\xi$, the fraction field of A and a local ring of dimension 0.

7. $\mathcal{O}(U) = \bigcap_{f \in x} A_f$. Obviously $A \subseteq \mathcal{O}(U)$. The claim is that $\mathcal{O}(U) = A$ Let $\frac{F}{G} \in k(x,y)$ with $G(0,0) = 0$. By a linear change of coordinates, we can assume that $y \nmid G$. Suppose that $\frac{F}{G} \in A_y$ so that $\frac{F}{G} = \frac{F'}{G'y^n}$ with $G'(0,0) \neq 0$. Then $FG'y^n = F'G$. Since $y \nmid G$, $y^n \mid F'$, so the y^n terms cancel in $\frac{F'}{G'y^n} = \frac{F}{G}$ so that $\frac{F}{G}$ is regular at $(0,0)$. Thus $\frac{F}{G} \in A$, so $D(U) = A$.

1.3 Schemes

1. Let $(X, \mathcal{O}_X)$ be a ringed space and G a group of automorphisms on it. Let $Y = X/G$, and $p : X \to Y$ the quotient map. To prove the first claim, we need to show there is an induced map from A^G to B^G in the following commutative diagram:

$$\begin{tikzcd}
A \arrow[d, "\rho"] & A^G \arrow[l] \arrow[d, dashed] \\
B & B^G \arrow[l]
\end{tikzcd}$$

. I.e we need to show that $g \cdot \rho(a) = \rho(g \cdot a) = \rho(a)$.

But this follows from the commutative diagram on sheaves for open $V \subset U$:

$$\begin{tikzcd}
\mathcal{O}_X(p^{-1}(U)) \arrow[d, "\rho"] & \mathcal{O}_X(p^{-1}(U)) \arrow[l, "g"'] \arrow[d, "\rho"] \\
\mathcal{O}_X(p^{-1}(V)) & \mathcal{O}_X(p^{-1}(V)) \arrow[l, "g"']
\end{tikzcd}$$

. So we have a presheaf which we now have to show is a sheaf. If we have some g-invariant sections s_i defined on some open sets U_i of Y, by looking at their pre-images in X, we get a global section s on X that restricts to each s_i. But then $g \cdot s$ locally look like $g \cdot s_i$, which is s_i, so $g \cdot s = s$

2. The topological equivalence of $\mathbb{P}^1$ and $Y = X/G$ is clear. The non-empty open sets of $\mathbb{P}^1$ are $\mathbb{P}^1$ and $\mathbb{P}^1$ minus a finite number of points. $\mathcal{O}_Y(\mathbb{P}^1) = \mathcal{O}_X(\mathbb{A}^2 \backslash 0)^G = k[\mathbb{A}^2]^G = k = \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)$ and $\mathcal{O}_Y(\mathbb{P}^1 \backslash \{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}) = \mathcal{O}_X(\mathbb{A}^2 \backslash$ lines through $\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\})^G = \{\frac{P(x,y)}{Q(x,y)} \mid P, Q$ homogeneous of deg 0 and $Q \neq 0\} \cong \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1 \backslash \{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\})$, so $(Y, \mathcal{O}_Y)$ and $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ are equivalent.

3. Y is a ringed space by ex 1. Let $(x,y) \in Y$ and suppose that $y \neq 0$. Let U be the open subset $Y \backslash (1,0)$. Then every element of U is uniquely expressed as $(a, 1)$ with $a \in k$. So let a denote $(a,1) \in U$. Functions invariant under the action of G are functions of xy. If a function of xy vanishes on the x-axis, it vanishes on the y-axis also, so $\mathcal{O}_Y(U) = k[xy]$, (that is, no denominators). Thus $\mathcal{O}_Y(U) \cong \mathcal{O}_{Spec\, k[t]}(\text{Spec } k[t]) \cong k[t]$. The isomorphism maps $xy \mapsto t$. And

$\mathcal{O}_Y(U \backslash \{a_1, \ldots, a_n\}) \cong \{\frac{p(xy)}{q(xy)} \,|\, q(a_i) \neq 0\} \cong \{\frac{p(t)}{q(t)} \,|\, q(a_i) \neq 0\} \cong \mathcal{O}_{Spec\, k[t]}($Spec $k[t] \backslash \{(t-a_1), \ldots, (t-a_n)\})$. These isomorphisms commute with the restriction maps and therefore $U \cong$ Spec $k[t]$. Thus (x, y) has affine neighborhoods. Similarly, (x, y) has an affine neighborhood if $x \neq 0$, so Y is a scheme. Now suppose that $Y = \mathbb{A}^2/G$. Any open set containing the origin intersects the x-axis, hence in a dense open subset, thus at some point $(x, 0)$, with $x \neq 0$. $\overline{0} \in U$ implies that $\overline{x} \in U$. Therefore x is not a closed point. But $\overline{x}$ has an affine neighborhood $\cong$ Spec $k[t]$ such that the prime ideal corresponding to $x = (t)$, which is maximal. Thus $\overline{x}$ is a closed point. Contradiction.

4. $\varphi : \mathbb{Z} \hookrightarrow \mathbb{Z}[i]$, φ^a : Spec $\mathbb{Z}[i] \to$ Spec $\mathbb{Z}$. Let p be prime. Then $\{(p)\} \cong$ Spec $\mathbb{Z}/p\mathbb{Z}$, so the inverse image is Spec $(\mathbb{Z}[i]/p\mathbb{Z}[i])$. If $p \equiv 1 \bmod 4$, then $\mathbb{Z}[i]/p\mathbb{Z}[i] = \mathbb{Z}[i]/(a+bi)(a-bi)\mathbb{Z}[i] \cong \mathbb{Z}[i]/(a+bi)\mathbb{Z}[i] \oplus \mathbb{Z}[i]/(a-bi)\mathbb{Z}[i]$, which has two prime ideals, both are maximal. If $p \not\equiv 1 \bmod 4$, $p\mathbb{Z}[i]$ is maximal, so the inverse image has one point, the zero ideal of $\mathbb{Z}[i]/p\mathbb{Z}[i]$.

5. This map is induced by the natural map $\lambda : \mathbb{R}[x] \to \mathbb{R}[x, y]/(x^2 + y^2 - 1) = A$. The prime ideals of $\mathbb{R}[x]$ are $(x - \alpha)$ and $(x^2 + \alpha x + \beta)$ with $x^2 + \alpha x + \beta$ irreducible. $A/\lambda(x - a) \cong \mathbb{R}[x, y]/(x - \alpha, x^2 + y^2 - 1) = \mathbb{R}[x, y]/(y^2 - 1 + \alpha^2, x - \alpha) \cong \mathbb{R}[y]/(y^2 - 1 + \alpha^2)$. If $-1 < \alpha < 1$, this is isomorphic to $\mathbb{R} \oplus \mathbb{R}$, so has a spectrum with 2 closed points. If $\alpha = \pm 1$, this is isomorphic to $\mathbb{R}[y]/(y^2)$, which has spectrum with one point. If $|\alpha| > 1$, this is isomorphic to $\mathbb{C}$, which has a one point spectrum. $A/\lambda(x^2 + \alpha x + \beta) \cong \mathbb{R}[x, y]/(x^2 + \alpha x + \beta, x^2 + y^2 - 1) \cong \mathbb{C}[y]/(y^2 - 1 + x^2)$, where x is now a complex root of $T^2 + \alpha T + \beta$. Obviously, $x \neq \pm 1$ since $T^2 + \alpha T + \beta$ is irreducible. So $-1 + x^2$ has distinct square roots. So the ring is isomorphic to $\mathbb{C} \oplus \mathbb{C}$, which has spectrum with two closed points.

6. Let $x \in \widetilde{X}$. Pick an open affine neighborhood $U =$ Spec $k[Y]$ of x with $Y \subseteq X$ an affine variety. Then x is a closed point iff x is a maximal ideal of $k[Y]$ iff x is a point of $Y \subseteq X$.

7. See Hartshorne pg 76-77

8. See Hartshorne Prop 2.5c pg 77

9. Proj $\Gamma = \bigcup D_+(x_i)$, where $D_+(x_i) =$ Spec $\Gamma_{(x_i)} =$ Spec $\{\frac{F}{x_i^n} \,|\, \deg F = n\} =$ Spec $A[\frac{x_o}{x_i}, \ldots, \frac{x_n}{x_i}] =$ Spec A_{x_i}. $D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j) =$ Spec $\Gamma_{(x_i x_j)} =$ Spec $A[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}] =$ Spec $A[x_1, \ldots, x_n]_{(x_i x_j)} =$ Spec $A[x_1, \ldots, x_n]_{(x_i)} \cap$ Spec $A[x_1, \ldots, x_n]_{(x_j)}$, so the gluing of the $D_+(x_i)$ to form Proj Γ is the same as the gluing of the A_i to form $\mathbb{P}^n_A$. So the identity maps $A_{x_i} \to \Gamma_{(x_i)}$ induce isomorphisms Spec $\Gamma_{x_i} \to$ Spec A_{x_i}, which glue together to form ~~Voltron~~ an isomorphism Proj $\Gamma \cong \mathbb{P}^n_A$.

10. See Hartshorne ex 7.12.1 on pg 163.

11. See Hartshorne ex 2.3.2 on pg 74

12. The ψ_U is just the push of $\mathcal{O}_X$ on Y, ie just the sheaf $\varphi_*\mathcal{O}_X$ on Y, so it defines the required homomorphism on stalks. The last statement follows from the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r] \arrow[d] & \mathcal{O}_X(\varphi^{-1}(U)) \arrow[d] \\
\mathcal{O}_{Y,y} \arrow[r] & \mathcal{O}_{X,x}
\end{tikzcd}$$

and the fact that $a \in \mathcal{O}(V)$ satisfies $a(z) = 0$ iff the image of a in the stalk is in the maximal idea. Thus the bottom row of the commutative diagram takes $m_{Y,y}$ into $m_{X,x}$ iff the top row takes elements vanishing at Y to elements which vanish at x.

1.4 Products of Schemes

1. For any scheme Z over an algebraically closed field k, closed points of Z correspond bijectively to morphisms Spec k → Z. By the universal property of the product, morphisms Spec k → $X \times_k Y$ correspond bijectively to pairs of morphisms (Spec k → X, Spec k → Y). Thus, closed points of $X \times_k Y$ correspond bijectively to pairs of closed points of X and Y.

2. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}[x]/(x^2+1) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[x]/(x^2+1) \cong \mathbb{C}[x]/(x+i) \oplus \mathbb{C}[x]/(x-i) \cong \mathbb{C} \oplus \mathbb{C}$. So Spec $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ consists of two points, both closed, with residue field $\mathbb{C}$.

3. $X \times_S X =$ Spec $(A \otimes_B A)$, so the existence of the induced map is clear. The conditions in 4.12 become:

1. $[(\varphi \circ \epsilon) \cdot 1_A] \circ \mu = [1_A \cdot (\varphi \circ \epsilon)] \circ \mu = 1_A$ defined by $(f \cdot g)(a_1 \otimes a_2) = f(a_1)g(a_2)$
2. $(i \cdot 1_A) \circ \mu = (1_A \cdot i) \circ \mu = \varphi \circ \epsilon$
3. $(\mu \otimes 1_A) \circ \mu = (1_A \otimes \mu) \circ \mu$ defined by $(f \otimes g)(a_1 \otimes a_2) \mapsto f(a_1) \otimes g(a_2)$

4. Let $\underline{chak = p}$ [sic]. The Frobenius map is induced by $\lambda : k[X] \to k[X]$ defined by $x \mapsto x^p$. The ideal of 0 is (x) and $\lambda(x) = (x^p)$, so $f^{-1}(0)$ is Spec $k[x]/(x^p)$.

$$\begin{tikzcd}
Spec\, k[X]/(x^p) \times_k Spec\, k[X]/(x^p) \arrow[r, "{(j,j)}"] \arrow[dr, "\mu'"] & Spec\, k[X] \times_k Spec\, k[X] \arrow[r, "\mu"] & Spec\, k[X] \, . \\
& Speck\, k[X]/(x^p) \arrow[ur] &
\end{tikzcd}$$

The associated ring homomorphisms are then:

$$\begin{tikzcd}
k[X]/(x^p) \otimes_k k[X]/(x^p) & k[X] \otimes_k k[X] \arrow[l, "q \otimes q"'] & k[X] \arrow[l, "x\otimes 1 + 1 \otimes x \leftarrow x"'] \arrow[dl, "q"] \\
& k[X]/(x^p) \arrow[ul, "x\otimes 1 + 1 \otimes x \leftarrow x"] &
\end{tikzcd}$$

The maps of ex 3 are then:

1. $\mu' : k[X]/(x^p) \to k[X]/(x^p) \otimes_k k[X]/(x^p)$ defined by $x \mapsto x \otimes 1 + 1 \otimes x$
2. $\epsilon^{'} : k[X]/(x^p) \to k$ defined by $x \mapsto 0$
3. $i : k[X]/(x^p) \to k[X]/(x^p)$ defined by $x \mapsto -x$

5. $\lambda : k[x, \frac{1}{x}] \to k[x, \frac{1}{x}]$ defined by $x \mapsto x^p$. $f^{-1} =$ Spec $k[x, \frac{1}{x}]/(x-1)^p$. The corresponding maps are:

1. $\mu' : k[x, \frac{1}{x}]/(x-1)^p \to k[x, \frac{1}{x}]/(x-1)^p \otimes_k k[x, \frac{1}{x}]/(x-1)^p$ defined by $x \mapsto x \otimes x$
2. $\epsilon' : k[x, \frac{1}{x}]/(x-1)^p \to k$ defined by $x \mapsto 1$
3. $i : k[x, \frac{1}{x}]/(x-1)^p \to k[x, \frac{1}{x}]/(x-1)^p$ defined by $x \mapsto \frac{1}{x}$

Spec $k[x, \frac{1}{x}]/(x-1)^p$ and Spec $k[X]/(x^p)$ each have only one point so there is a unique set map between them. An isomorphism would induce an isomorphism on local rings so the maximal ideals of the local rings are taken to one another. Thus an isomorphism is a local morphism of ringed spaces, thus by 1.3 Thm 1 is of the form λ^a with $\lambda : k[x]/(x^p) \to k[x, \frac{1}{x}]/(x-1)^p$ an isomorphism. But then λ^{-1} would take x and $\frac{1}{x}$ into k, so it isn't surjective.

6. These are called dual numbers. They are annoying.

7. Let Y be the scheme of ex 3.3. We saw in ex 3 that $Y \backslash \{x\text{- axis }\} \cong$ Spec $k[t]$ and $Y \backslash \{\text{y}-axis\} \cong$ Spec $k[t]$. The points of the two Specs are glued together the same way as the bug-eyed affine line since the equivalence class of $(1,t)$ is the class of $(t,1)$ if $t \neq 0$, meaning that the isomorphism of open sets Spec $k[t, \frac{1}{t}] \to$ Spec $k[t, \frac{1}{t}]$ is induced by the identity map on $k[t, \frac{1}{t}]$. The bug eyes are the points corresponding to the two axes.

8. Assume that Γ is finitely generated over Γ_0. Proj Γ is a projective scheme, and any projective scheme is separated since any projective scheme is proper by Hartshorne Thm 4.9 p 103.

2 Ch. VI: Varieties

2.1 Definitions and Examples

1. Nick would like to not answer this problem on a technicality. There should be a separatedness assumption since the bug-eyed affine line is a pseudo-variety but not a variety.

2. Let $X \times Y = \{(x,y) \,|\, x \in X, y \in Y\}$. If $U_\alpha \ni x$ and $V_\beta \ni y$ are neighborhoods isomorphic to m-Spec of a ring (ie some affine variety), then $U_\alpha \times V_\beta$ is isomorphic to an affine variety by book 1.

3. Let X be a complete variety and let $p_2 : X \times Y \to Y$ be the second projection which takes closed sets to closed sets for any variety Y. Let $X = \bigcup X_i$ be the decomposition of X into a finite number of irreducible components. Then since $X_i \times Y$ is a closed subset of $X \times Y$, and $X \times Y = \bigcup (X_i \times Y)$, the restriction of p_2 to $X_i \times Y$ then takes the closed set to another closed set in Y, so X_i is proper. Conversely, if X_i is proper for all i, then for any closed set $C \subseteq X$, $C = \bigcup (C \cap X_i)$. So $p_2 : X \times Y \to Y$ sends $C \mapsto p_2(C) = \bigcup p_2(C \cap X_i)$ which is a union of closed sets since each X_i is complete, so X is also.

4. Let $X \to S$ be a locally trivial fibration with S and F both complete. Using the definition for a complete variety as a variety proper over Spec k, and since proper is local on the base, we get that $F \times U$ is proper over U by base extension of the proper map $F \to$ Spec k in the commutative diagram:

$$\begin{tikzcd}
& F \times U \arrow[dl] \arrow[dr, "prop?\ \checkmark"] & \\
F \arrow[dr, "prop"'] & & U \subseteq S \arrow[dl] \\
& Spec\, k &
\end{tikzcd}$$

So $X \to S$ is proper since $X|_U \cong F \times U \to U \subseteq S$ is proper. Finally $X \to$ Spec k is proper over Spec k (ie complete) since it is the composition of two proper maps $X \to S$ and $S \to$ Spec k.

5. In the piece $X_\alpha \neq 0$, the map is $(x_0, \ldots, x_n; y_0, \ldots, y_n) \mapsto (\frac{x_0}{x_\alpha}, \ldots, \frac{x_n}{x_\alpha}; y_\alpha)$. For $X_\beta \neq 0$, the map is $(x_0, \ldots, x_n; y_0, \ldots, y_n) \mapsto (\frac{x_0}{x_\beta}, \ldots, \frac{x_n}{x_\beta}; y_\beta)$. So the transition matrix from α to β is $G_{\alpha\beta} = \frac{y_\beta}{y_\alpha} = \frac{x_\beta}{x_\alpha}$ since $x_\alpha y_\beta = x_\beta y_\alpha$. This line bundle is dual to that of 1.4 ex 2 so it's corresponding divisor is $-E$, where E is the hyperplane section.

6. Assume that X is normal and that D has no fixed componets. Let $\mathcal{L}(D) = \text{Span}\{f_0, \ldots, f_n\}$, where each $f_i \in k(X)$ and $(f_i) + D \geq 0$. Then x is not a basepoint of the linear system $|D|$ iff the corresponding morphism $\varphi_{|D|} : X \to \mathbb{P}^{dim\,|D|}$ defined by $x \mapsto (f_0(x), \ldots, f_n(x))$ is regular at x iff there exists a $g \in k(X)$ such that $f_i g(x) \neq 0$ for some i and the rest have to regular at x.

Let $f_0, \ldots, f_n$ generate the stalk of the sheaf $\mathcal{F}_x = \frac{1}{g}\mathcal{O}_{U,x}$, where g is the local equation of the divisor at x. Moving the g over, we get that gf_i generate the stalk $\mathcal{O}_{U,x}$. Since $\mathcal{O}_{U,x}$ is a local ring, this is the same as saying that for some f_i, say f_0, $gf_0(x)$ is nonzero, ie a unit, ie $gf_0 \notin m_{U,x}$, the maximal ideal of $\mathcal{O}_{U,x}$. So then $(gf_0(x), \ldots, gf_n(x)) = (f_0(x), \ldots, f_n(x))$ is a well-defined point of $\mathbb{P}^n$. So the map $\varphi_{|D|}$ is regular. Reverse the arguments above to give the converse implication.

7. Let $X = \text{Spec } A$ be a nonsingular affine variety. Then the sheaf of relative differentials $\Omega_{X/k}$ is the module of relative differentials squiggled, ie $\widetilde{\Omega_{A/k}}$. The dual of this sheaf is the tangent sheaf, $\mathcal{T}_{X/k} = \text{Hom}_A(\Omega_{A/k}, A)^\sim = \text{Der}_k(A, A)^\sim$ by the universal property of $\Omega_{A/k}$. Taking global sections then gives that $\Theta_X(X) = \Gamma(X, \mathcal{T}_{X/k}) = \text{Der}_k(A, A)$ as desired.

8. Pick $C = \{x_2 = \ldots = x_n = 0\}$. C is covered by $U_0 = \{x_0 \neq 0\}$ and U_1. Let $U_i = \frac{x_i}{x_0}$ and $v_1 = \frac{x_0}{x_1}$, $v_i = \frac{x_i}{x_1}$ for $i = 1, \ldots n$ be local parameters after subtracting constants at each point of U_0 and U_1. Using the formula below formula (9) on pg 63, with $i, j = 2, \ldots, n$ since $u_2, \ldots, u_n$ are local equations of C, not $u_1, \ldots, u_{n-1}$ like in the example. Same with $v_2, \ldots v_n$. Noting that $v_i = \frac{u_i}{u_1}$ for $i > 1$, we get that C_{01} is a diagonal matrix with $\frac{1}{u_1}$'s along the diagonal. So $N_{\mathbb{P}^n/C}$ is a direct sum of a line bundle E which is trivial on U_0 and U_1 and has transition matrix $C_{01} = [\frac{1}{u_1}] = [\frac{x_0}{x_1}]$. So $E = \mathcal{O}(1)$ as in example 2, pg 66 applied to $C \cong \mathbb{P}^1$. Thus $c(E)$ is the hyperplane section.

9. Since the intersection is transversal, the local equations of the hyperplanes C_i may be included in a system of local parameters at any point of X. Let $\{U_\alpha\}$ be a cover of $\mathbb{P}^n$ such that the local equation $f_{i,\alpha}$ of C_i together with some function g_α form a system of local parameters at every point of $U_\alpha \cap X$. In the intersection $U_\alpha \cap U_\beta$, we have $f_{i,\beta} = \frac{f_{i,\beta}}{f_{i,\alpha}} f_{i,\beta}$. So the transition matrix $C_{\alpha\beta}$ of the normal bundle $N_{X/\mathbb{P}^n}$ is a diagonal matrix of the form $C_{\alpha\beta} = (f_{i,\beta}/f_{i,\alpha})$ by page 9 on page 63. The transition matrix of det $N_{X/\mathbb{P}^n}$ is then $[\det C_{\alpha\beta}] = \prod f_{i,\beta}/f_{i,\alpha}$. So c(det $N_{X/\mathbb{P}^n}) = \rho_x(c_1) + \ldots + \rho_x(c_{n-1})$. $K_{\mathbb{P}^n} = (-n-1)H$, where H is the hyperplane divisor, so deg $\rho_x(K_{\mathbb{P}^n}) = (-n-1)\text{deg}X$. By the adjunction formula (8) on page 67, deg $c(\Omega_x) =$ deg $\rho_x(c(\Omega^n_{\mathbb{P}^n}))+$ deg $c(N_{x/\mathbb{P}^n}) = 2g - 2 = (-n-1)$deg $X+$ deg $\rho_x(C_1) + \ldots +$ deg $\rho_x(C_{n-1}) = (-n-1)\prod k_i + k_1 \text{ deg}X + \ldots + k_{n-1}$deg X$= (-n-1)\prod k_i + \sum k_i \prod k_i$. Solving for g gives $g = \frac{1}{2}\prod k_i(-n-1+\sum k_i)+1$.

10. This exercise is trying to explain the Projective Space Bundle of a locally free vector bundle E. Considering E as a locally free sheaf $\mathcal{E}$, this follows from Hartshorne Definition/Propositon 7.11 on page 162.

11. This is Thm V.2.17 in Hartshorne p 379, with the notation from Shaf to Hartshorne as follows: n is e, $C_\infty = C_0$, C_0 is C_1. 12. Viewing the projective space bundle as a ruled surface, then this follows from Hartshorne Prop V.2.3. p 370

13. Similar as above, this is done in Hartshorne Lemma V.2.10 on page 373.

14. Once again, this is the classification of rational ruled surfaces, done in Hartshorne Cor V.2.13 p 376.

2.2 Abstract and Quasiprojective Varieties

1. Let X be a nonsingular, irreducible variety and $f : X \to Y$ a morphism from X to a complete variety Y. By Chow's lemma, there is a projective variety Z and a birational morphism $g : Z \to Y$. By Chap II, 3.1, Theorem 3, the composition $g^{-1}f$ has locus of indeterminacy of codimension ≥ 2. Thus the same is true of $gg^{-1}f = f$.

2. Let Y be a variety. By the middle of pg 79, the map $\sigma \times 1 : X^{'} \times Y \to X \times Y$ takes closed sets to closed sets. The projection $X \times Y \to Y$ takes closed sets to closed sets by completeness of X, so the composition $X^{'} \times Y \to Y$ takes closed sets to closed sets. Hence, $X^{'}$ is complete.

3. Choose an open covering of X where both $\mathcal{E}$ and $\mathcal{L}$ are free. The transition matrices of $\mathcal{E} \otimes \mathcal{L}$ are the transition matrices of $\mathcal{E}$ with each entry multiplied by the transition functions of $\mathcal{L}$. But for projective linear transformations, multiplying each entry of a matrix by the same element changes nothing. So the corresponding projective space bundles $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ have the same transition matrices and hence are isomorphic.

4. Move the support of D away from y_0. Then the support of $\sigma^*(D)$ and l are distinct so $\sigma^*(D)l = 0$.

5. Let D be a surface containing Y and nonsingular at y_0. Write $\sigma^*(D) = S + F$, where F is the birational transform of D. D has multiplicity 1 at y_0 so $F.l = 1$. Thus $S.l = (\sigma^*(D) - F).l = 0 - 1 = -1$.

6. Let $X^{'}$ be the blowup of X at some point. Then $X^{'}$ contains a copy of $\mathbb{P}^2$, namely the exceptional divisor. In this $\mathbb{P}^2$ choose a line and a conic meeting at two distinct points. Now apply the construction of Hironaka's counterexample to get a complete nonsingular nonprojective variety birational to $X^{'}$ and hence birational to X.

7. By Hartshorne pg 105, X can be embedded as an open dense subset of a complete variety Y. Apply Chow's lemma to Y to get a birational morphism $\varphi : Z \to Y$ with Z projective. Then $\varphi : \varphi^{-1}(X) \to X$ is a birational morphism from a quasiprojective variety onto X.

2.3 Coherent Sheaves

1. Since the question is local in nature, we can assume that $X = \text{Spec } A$. with A Noetherian and $\mathcal{F} = \widetilde{M}$. If $\mathcal{F}$ is a torsion sheaf, then $M = Ax_1 + \ldots + Ax_n$ is a torsion module. Pick $f_i \in A$ such that $f_i x_i = 0$. Then $f_1 \cdots f_n$ annihilates M, So $M_{f_1 \cdots f_n} = 0$. So $\mathcal{F}|_{D(f_1 \cdots f_n)} = 0$ and thus the stalk of $\mathcal{F}$ vanishes at the

generic point of X. So $\mathcal{F}$ has support distinct from X. Conversely, if $\mathcal{F}$ has support distinct from X, then there exists $f \in A$ such that $\mathcal{F}|_{D(F)} = 0$ so that $M_f = 0$. Thus for all $m \in M$, there exists n such that $f^n m = 0$, so M is a torsion module, so $\mathcal{F}$ is a torsion sheaf.

2. If $X = \text{Spec } A$ is a nonsingular affine curve, then A is a Dedekind Domain. A torsion sheaf $\mathcal{F}$ is of the form $\widetilde{M}$, where M is a finite torsion A-module. By the structure theorem for finite modules over a Dedekind Domain, $M \cong A/p_1^{e_1} \oplus \ldots \oplus A/p_n^{e_n}$ (D and F, 16.3). If $f \in \bigcap p_i$, then $M_f = 0$, so $\mathcal{F}(D(f)) = 0$. Otherwise $M_f \neq 0$. So Supp $\mathcal{F} = \{x_1, \ldots, x - n\}$ where x_i is the closed point corresponding to p_i. In the general case for an abstract curve, glue like hell.

3. The sheaf $\mathcal{L}_E$ of sections of E is a coherent sheaf hence is of the form $\widetilde{M}$. $\widetilde{M}(X)$ is a finite $\mathcal{O}_X(X)$-module, but $M_E = \widetilde{M}(X)$ is the set of global section and $\mathcal{O}_X(X) \cong A$. 4. Let m be the maximal ideal of A. Then There exists $U \ni x$ such that $\widetilde{M_E}|_U$ is free over $\mathcal{O}_X|_U$, so the same holds for the local rings at x, ie $(M_E)_x$ is a free A_x-module. So by Matsumura, Thm 7.12 pg 52, M_E is a projective A-module.

5. $E \to X$ and $E^{'} \to X$ are isomorphic vector bundles iff $\mathcal{L}_E \cong \mathcal{L}_{E'}$ iff $\widetilde{M_E} \cong \widetilde{M_{E'}}$, where $\mathcal{L}_E \cong \widetilde{M_E}$, $\mathcal{L}_{E'} \cong \widetilde{M^{'}_E}$, and $M_E, M_{E'}$ are finite A-modules iff $M_E \cong M_{E'}$.

6. This is equivalent to the statement that every locally free sheaf of finite rank on $\mathbb{A}^1$ is free. A locally free sheaf $\mathcal{E}$ of finite rank is coherent so is isomorphic to $\widetilde{M}$ for some finitely generated $k[x]$-module M. $\mathcal{E}$ is locally free so M is locally free, hence projective. By the structure theorem for finitely generated modules over P.I.Ds, a finitely generated projective module over a P.I.D. is free. Hence M is free, and so the same is true of $\mathcal{E}$.

7. The sheaf of sections of a vector bundle of finite rank is a locally free sheaf of finite rank, hence coherent. The result then follows from the theorem of 3.4.

8. By the equivalence of the categories of vector bundles and locally free sheaves of finite rank we may instead show that the space of $\mathcal{O}_X$-module morphisms $\mathcal{E}_1 \to \mathcal{E}_2$ is finite dimensional, where $\mathcal{E}_1$ and $\mathcal{E}_2$ and locally free of finite rank. This space is the space of global sections of the coherent sheaf $\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)$, hence is finite-dimensional by the theorem of 3.4.

9. L must be M_x, the localization of M at the prime ideal $x = (0)$ of A. φ must be the localization map $M \to M_x$. Consider the subsheaf $\mathcal{F} \subseteq \mathcal{O}_X$ given by $\mathcal{F}(X) = 0$ and $\mathcal{F}(U) = K$. X has no nontrivial open coverings so the sheaf axiom doesn't need to be checked. $\mathcal{F}$ clearly isn't coherent since $0_x \ncong K$.

10. The sheaf axiom is clearly satisfied away from x_0. So let $x_0 \in U$ and let $\{U_i\}, i \in \Lambda$ be an open covering of U with $x_0 \in U_0$. Let $\{s_i \in F(U_i)\}, i \in \Lambda$ be a set of compatible sections. $s_0 = 0$ since $F(U_0) = 0$. By compatibility of the s_i, each s_i must be 0 on the dense open subset $U_i \cap U_0$ of U_i, hence $s_i = 0$ for all $i \in \Lambda$. Thus the unique lift of $\{s_i\}$ in $\mathcal{F}(U)$ is 0. So $\mathcal{F}$ is a sheaf. Now let U be an open affine subset of X containing x_0 and let $V \subseteq U$ be an affine open subset of U not containing x_0. If $\mathcal{F}$ were coherent, then we'd have $\mathcal{F}(V) \cong \mathcal{F}(U) \otimes_{k[U]} k[V]$. But $\mathcal{F}(U)$ is 0 and $\mathcal{F}(V)$ is nonzero.

2.4 Classification of Geometric Objects and Universal Schemes

1. This is Thm 11.1 (Hilbert-Serre) on page 117 in Atiyah-MacDonald. .

2. This follows from the bottom of pg 52 of Hartshorne which says that the Hilbert Polynomial for X is $P_H(z) = \binom{z+n}{n} - \binom{z-d+n}{n}$. $a_r(X) = P_X(r)$ for all $r > d$.

3. See Hartshorne bottom of page 52.

4. Let $\{f = 0\}$ be the hypersurface of degree d. The argument in the proof of 4.2 Thm 2 shows that $k = 0$ and $a_{X'} = (f, a_x)$, so $C = S/(f, a_X) = S/a_{X'}$.. SO the sequence 4.2(1) becomes, (with $M = S/a_X$) and ξ_n replaced with f: $0 \to S^{(r)}/a_X^{(r)} \xrightarrow{f} S^{(r+d)}/a_X^{(r+d)} \to S^{(r+d)}/(f, a_x) = S/a_X^{'} \to 0$. So $P_X^{'}(T+D) = P_X(T+d) - P_X(T)$.

5. Using ex 3 and 4, we get that $P(T + d_2) = \binom{n+t+d_2}{n} - \binom{n+T+d_2-d_1}{n} - \binom{n+T}{n} + \binom{n+T-d_1}{n}$.

6. Assume they mean $F'(0) = 0$, else this ring is just k. Call this ring A. let $a = (T^2, T^3) \subseteq A = k[T^2, T^3]$. Assume that $T^2 \otimes T \neq T^3 \otimes 1$ in $a \otimes_A B$. Then $T^2 \otimes T - T^3 \otimes 1 \mapsto 0$ under $a \otimes_A B \to aB$. So B is not flat over A.

7. See Atiyah-MacDonald, Cor 3.6 pg 40.

8. Let B be the homogeneous coordinate ring of X. Then B is a quotient of $k[x_0, \ldots, x_n] \cong k[\mathbb{A}^{n+1}]$, so Spec B is closed subscheme of A^{n+1}.

9. The fibers with $c(t) \neq 0$ are isomorphic by Ex 3.3.8.

10. I'm going to cheat and do this the short way. Nick is giving me a severe look of disapproval and calling me names. Ow. Now he is hitting me. Using Hartshorne ex 7.2 on page 54, we know that the arithmetic genus determines the constant term of the Hilbert polynomial. For the case of the plane conic, which is birational to $\mathbb{P}^1$, its genus is 0, so the constant term $P_H(0) = 1$. Using Thm 2 on pg 103, we know that the degree of each Hilbert polynomial is 1, and

the leading term is dT^n. So the Hilbert Polynomial of the conic is just $2T + 1$ and the Hilbert Polynomial of the 2 skew lines is just the sum of the Hilbert Polynomials for each one, which is just $T + 1$, so the final Hilbert Polynomial of the 2 skew lines is $2T + 2$.

11. By the proof of 4.3 Thm 3, the Hilbert Polynomials being the same is equivalent to $S(X)^{(r)}$ being torsion free for $r >> 0$, where $S(X)$ is the homogeneous coordinate ring of X. Let n be such that $S(X)^{(r)}$ has no torsion for $r \geq n$. See also Hartshorne Thm 9.9 pg 261 for generalization.