This exam contains 11 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

1. **Show your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

2. **Follow the instructions closely**. For example, if you are asked to justify your answers, then do so in a brief and coherent way.

3. **Points will be taken off for incorrect statements, even if correct ones are present**. Be careful about what you include in your answers. If they contain both the correct answers and incorrect or nonsense statements, points will be taken off.

4. If you need more space, use the back of the pages; clearly indicate when you have done this.

**Good luck!!** Do not write in the table to the right.
1. (20 points) Consider the vectors

\[
\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

(a) (5 points) Show that the vectors in the set \(B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}\) are linearly independent.

(b) (5 points) Find the representation of \(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\) in the \(B\) basis, i.e. find

\[
\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_B
\]

with \(B\) as in the previous part. (Express this vector as a linear combination of \(\vec{v}_1, \vec{v}_2, \vec{v}_3\), and...)}
(c) (10 points) Consider the linear map \( \text{Ref}_P : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) which takes a vector \( \vec{x} \) to its reflection over the plane \( \{x = y\} \). Find a matrix \( B \) for \( \text{Ref}_P \) in the basis \( B = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \} \) from the previous part.

Hint: it’s easiest to get \( B \) directly by checking what \( \text{Ref}_P \) does to the vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) above (rather than trying to use a change of basis matrix.)
2. (20 points) Let $V$ be a linear space. Complete the following definitions.

(a) (8 points) 1) A set of elements $f_1, \ldots, f_n$ in $V$ is linearly independent if...

2) $V = \text{span}\{f_1, \ldots, f_n\}$ if...

3) $f_1, \ldots, f_n$ form a basis for $V$ if...

4) $\dim V = k$ if...

(b) (2 points) Let $\text{Poly}_4 = \{at^4 + bt^3 + ct^2 + dt + e : a, b, c, d, e \in \mathbb{R}\}$ be the space of polynomials of degree less than or equal to 4. What is $\dim \text{Poly}_4$?
(c) (10 points) Let $Poly_2 = \{at^2 + bt + c : a, b, c \in \mathbb{R}\}$ be the space of polynomials of degree less than or equal to 2. Consider the two bases

$\mathcal{A} = \{f_1 = 1, f_2 = t, f_3 = t^2\}, \quad \mathcal{B} = \{g_1 = 1, g_2 = t - 1, g_3 = t^2 - t\}.$

Compute $S_{\mathcal{A} \rightarrow \mathcal{B}}$ and $S_{\mathcal{B} \rightarrow \mathcal{A}}$. (Recall that $S_{\mathcal{A} \rightarrow \mathcal{B}}(f)_{\mathcal{A}} = (f)_{\mathcal{B}}$.)
3. (20 points) (a) (6 points) Consider the map

\[ T(f) = f'' + 4f' \]

for \( T : \text{Poly}_2 \rightarrow \text{Poly}_2, \)

where \( \text{Poly}_2 \) is the space of degree two polynomials from the previous problem. Compute the kernel, \( \ker(T) \). Is \( T \) an isomorphism? (An isomorphism is just an invertible linear map.)

(b) (4 points) For the map \( T \) in the previous part, is \( f(t) = t - 1 \) in \( \text{image}(T) \)?
(c) (5 points) For $T$ as in the previous part, compute the matrix of $T$ in basis $\mathcal{A} = \{f_1 = 1, f_2 = t, f_3 = t^2\}$. That is, find a matrix $A$ for which

$$A(f)_{\mathcal{A}} = (T(f))_{\mathcal{A}}.$$ 

(d) (5 points) True or false: the matrix $A$ from the previous part satisfies $\text{image}(A) = \mathbb{R}^3$. Justify your answer.
4. (20 points) Consider the transpose map \( T: \text{Mat}_{2 \times 2} \rightarrow \text{Mat}_{2 \times 2} \), where \( \text{Mat}_{2 \times 2} \) is the space of \( 2 \times 2 \) matrices and

\[
T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}
\]

(a) (8 points) Find the matrix for \( T \) in the basis

\[
\mathfrak{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.
\]
(b) (9 points) Find a basis $\mathcal{B}$ of $Mat_{2 \times 2}$ in which the matrix $B$ for $T$ the transpose map is diagonal: that is, such that the matrix $B$ satisfying

$$B(M)_{\mathcal{B}} = (T(M))_{\mathcal{B}}$$

has non-zero entries only on the diagonal. Compute $B$.

Hint: Find three independent matrices $M$ with $T(M) = M$ and one matrix with $T(M) = -M$.

(c) (3 points) What is the inverse of the transpose map, i.e. the map $T^{-1}$ with $T(M) = N$ if and only if $T^{-1}(N) = M$. 
5. (20 points) (a) (5 points) Complete the definition: \( \vec{u}_1, \ldots, \vec{u}_m \) in \( \mathbb{R}^n \) form an orthonormal set if...

(b) (5 points) Let \( u_1, u_2, u_3 \) be orthonormal vectors in \( \mathbb{R}^n \). Let

\[ \vec{x} = \vec{u}_1 + \vec{u}_2 + 2 \vec{u}_3. \]

Compute \( \|\vec{x}\| \).
(c) (10 points) The set

$$\mathfrak{B} = \left\{ \vec{u}_1 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix} \right\}$$

is an orthonormal basis of $\mathbb{R}^3$. (You don’t need to check this.) Find the change of basis matrix $S_{\mathfrak{A} \to \mathfrak{B}}$ where $\mathfrak{A} = \{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$ is the standard basis of $\mathbb{R}^3$. (Thus $S_{\mathfrak{A} \to \mathfrak{B}} \vec{v} = (\vec{v})_\mathfrak{B}$.)

Hint: note that $S_{\mathfrak{A} \to \mathfrak{B}} = S_{\mathfrak{B} \to \mathfrak{A}}^{-1}$. We discussed a mechanism for inverting matrices with orthonormal columns in class, and it is mentioned somewhere in this exam.