COMMUTING ELEMENTS IN CENTRAL PRODUCTS OF SPECIAL UNITARY GROUPS

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Abstract. In this paper the space of commuting elements in the central product $G_{m,p}$ of m copies of the special unitary group $SU(p)$ is studied, where p is a prime number. In particular, a computation for the number of path-connected components of these spaces is given and the geometry of the moduli space $\mathrm{Rep}(\mathbb{Z}^n, G_{m,p})$ of isomorphism classes of flat connections on principal $G_{m,p}$–bundles over the n–torus is completely described for all values of n, m and p.

1. Introduction

Let G be a compact Lie group. The space of homomorphisms $\mathrm{Hom}(\mathbb{Z}^n, G)$ can be identified with the space of commuting n-tuples in G, topologized as a subspace of the cartesian product G^n. The quotient $\mathrm{Hom}(\mathbb{Z}^n, G)/G$ under the conjugation action by G is the moduli space of isomorphism classes of flat connections on principal G-bundles over the n-torus $(\mathbb{S}^1)^n$. In the past few years there has been an increasing interest in understanding these spaces, especially in computing their number of path-connected components and their cohomology groups as they naturally appear in a number of quantum field theories such as Yang-Mills and Chern-Simons theories.

In [5] the space of commuting elements in a Lie group G was analyzed by considering the space of almost commuting elements in the universal cover of G (i.e. elements which commute up to central elements, see Definition 3). In particular it was shown that $\mathrm{Rep}(\mathbb{Z}^n, G) := \mathrm{Hom}(\mathbb{Z}^n, G)/G$ is determined by the geometry of G and explicit formulations were given for $n = 2$ and $n = 3$; indeed the main focus there was to describe the associated moduli spaces of bundles over $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$.

On the other hand, in [1] the spaces of the form $\mathrm{Hom}(\mathbb{Z}^n, G)$ were studied from a homotopical point of view. In particular, it was shown that if G is a closed subgroup of $GL(n, \mathbb{C})$, then there exists a natural homotopy equivalence after a single suspension

$$\Theta_n : \Sigma(\mathrm{Hom}(\mathbb{Z}^n, G)) \simeq \bigvee_{1 \leq r \leq n} \Sigma \left(\bigvee^{\binom{n}{r}} \mathrm{Hom}(\mathbb{Z}^r, G)/S_r(G) \right), \tag{1}$$

where $S_r(G) \subset \mathrm{Hom}(\mathbb{Z}^r, G)$ is the subspace of r-tuples $(x_1, ..., x_r) \in \mathrm{Hom}(\mathbb{Z}^r, G)$ for which at least one of the x_i equals 1_G. In [2], the authors show that a similar decomposition to (1) also holds for the space of almost commuting elements in a compact Lie group G and that the corresponding map Θ_n is actually a G-equivariant

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homotopy equivalence thus affording a stable decomposition for the associated spaces of representations.

Based on these stable homotopy equivalences it seems natural to explore situations where the geometric description of the moduli spaces associated to commuting pairs and triples provided in [5], can be extended to arbitrary commuting n–tuples. In particular it can be seen that if the maximal abelian subgroups in G are path-connected, then all of the spaces $\mathrm{Hom}(\mathbb{Z}^n, G)$ are path-connected. However, if the fundamental group of G has p–torsion, then it is known (see [4], page 139) that there is a subgroup $\mathbb{Z}/p \times \mathbb{Z}/p \subset G$ which is not contained in a torus and so the spaces of commuting n–tuples cannot be path–connected. Thus it is natural to consider examples where $\pi_1(G) \cong \mathbb{Z}/p$.

In this paper the spaces of the form $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})$ are studied, where

$$G_{m,p} = (SU(p)^m)/(\Delta(\mathbb{Z}/p))$$

is an m-fold central product of $SU(p)$, for a prime p. Thus these are natural examples of compact Lie groups having a fundamental group of prime order. The study of almost commuting elements in $SU(p)^m$ provides a way to compute the number of path-connected components of $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})$. In addition, the structure of the components can be explicitly described. The following theorem summarizes these results:

Theorem 1. *For $n \geq 1$ and p a prime number, the space $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})$ has*

$$N(n,m,p) = \frac{p^{(m-1)(n-2)}(p^n-1)(p^{n-1}-1)}{p^2-1} + 1$$

path–connected components. The path-connected component containing $(1,...,1)$ is a quotient of, and has the same rational cohomology as,

$$(G_{m,p}/(\mathbb{S}^1)^{m(p-1)}) \times_{(\Sigma_p)^m} (\mathbb{S}^1)^{m(p-1)n},$$

whereas all the other path–connected components are homeomorphic to

$$(SU(p))^m/((\mathbb{Z}/p)^{m-1} \times E_p),$$

where $E_p \subset SU(p)$ is the quaternion group Q_8 of order eight when $p=2$ and the extraspecial p–group of order p^3 and exponent p when $p>2$.

In section 3 it is explained how the path-connected component of $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})$ containing $(1,...,1)$ can be seen as a quotient of the compact manifold

$$(G_{m,p}/(\mathbb{S}^1)^{m(p-1)}) \times_{(\Sigma_p)^m} (\mathbb{S}^1)^{m(p-1)n}.$$

A particular case of relevance of Theorem 1 is the case where $m=1$. In this case, $G_{1,p} = PU(p)$ and according to the theorem $\mathrm{Hom}(\mathbb{Z}^n, PU(p))$ has

$$N(n,1,p) = \frac{(p^n-1)(p^{n-1}-1)}{p^2-1} + 1$$

path–connected components. Moreover,

$$\frac{(p^n-1)(p^{n-1}-1)}{p^2-1}$$

of these components are homeomorphic to $SU(p)/E_p$. On the other hand, the number x_n of path–connected components of $\mathrm{Hom}(\mathbb{Z}^n, SO(3))$ that *do not* contain the element $(1, ..., 1)$ was computed in [6], where it was shown that

$$x_n = \begin{cases} \frac{1}{6}(4^n - 3\times 2^n + 2) & \text{if } n \text{ is even,} \\ \frac{2}{3}(4^{n-1}-1) - 2^{n-1} + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Note that in Theorem 1 the case $p = 2$ and $m = 1$ corresponds to

$$G_{1,2} = SU(2)/(\mathbb{Z}/2) = PU(2) \cong SO(3)$$

which is precisely the situation already studied [6]. It is easy to verify that

$$x_n = \frac{(2^n-1)(2^{n-1}-1)}{3}$$

and thus the two approaches give the same answer.

Taking a quotient by the conjugation action of $G_{m,p}$ yields the following.

Theorem 2. *The moduli space of isomorphism classes of flat connections on principal $G_{m,p}$–bundles over an n–torus is given by*

$$\mathrm{Rep}(\mathbb{Z}^n, G_{m,p}) \cong ((\mathbb{S}^1)^{(p-1)mn}/(\Sigma_p)^m) \sqcup X_{n,m,p},$$

where $X_{n,m,p}$ is a finite set with $N(n, m, p) - 1$ points.

As can be expected, these quotient spaces are much simpler than the spaces of homomorphisms lying above them, which can contain interesting geometric information which is lost modulo conjugation; suffice it to say that for $n = 1$ this is the difference between the group $G_{m,p}$ and its quotient under conjugation T/W where $T \subset G_{m,p}$ is a maximal torus with Weyl group W. Also, it's worth noting that the components which do not correspond to the identity element deserve special attention, as they are somewhat exotic.

It also seems relevant to point out that the central products considered here arise as subgroups of some of the exceptional Lie groups. For example

$$G_{2,2} \subset \mathbb{G}_2,\ G_{2,3} \subset \mathbb{F}_4,\ G_{2,5} \subset \mathbb{E}_8,\ G_{3,3} \subset \mathbb{E}_6$$

and they give rise to subgroups of the form $(\mathbb{Z}/p)^3$ which are not contained in the maximal tori, thus explaining the torsion in the cohomology of the classifying spaces of these exceptional groups (see [4], pages 153–154) even though they are simply connected. It would seem that the results here could be applied to provide information about $\mathrm{Rep}(\mathbb{Z}^n, G)$, where G is one of these groups.

Notation. *From now on, for a prime number* p, E_p *denotes the quaternion group* Q_8 *of order eight when* $p = 2$ *and the extraspecial* p*–group of order* p^3 *and exponent* p *when* $p > 2$. *Note that this group can be identified with the* p*–Sylow subgroup of* $SL_3(\mathbb{F}_p)$. *Also, given an integer* $m \geq 1$,

$$G_{m,p} := (SU(p)^m)/(\Delta(\mathbb{Z}/p)),$$

here $\Delta(\mathbb{Z}/p)$ *is seen as a subgroup of* $SU(p)^m$ *by considering the diagonal map*

$$\Delta(\mathbb{Z}/p) \hookrightarrow (\mathbb{Z}/p)^m = Z(SU(p)^m).$$

Thus $G_{m,p}$ *is the* m*-fold central product of* $SU(p)$.

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2. Almost commuting elements

In this section almost commuting elements in a Lie group are introduced.

Definition 3. *Take* G *a Lie group and* $K \subset Z(G)$ *a closed subgroup. An* n*-tuple* $\underline{x} := (x_1, ..., x_n) \in G^n$ *is said to be a* K*-almost commuting* n*-tuple if* $[x_i, x_j] \in K \subset Z(G)$ *for every* $1 \leq i, j \leq n$.

The motivation for considering almost commuting elements is as follows. Consider the space $\mathrm{Hom}(\mathbb{Z}^n, H)$, where H can be written in the form $H = G/K$, for a Lie group G and a closed subgroup $K \subset Z(G)$. In this case, the natural map $f : G \to G/K$ is both a homomorphism and a principal K-bundle. If $\underline{x} = (x_1, ..., x_n)$ is a sequence of elements in G/K that commute, then for any lifting $\tilde{x}_i$ of x_i the commutator $[\tilde{x}_i, \tilde{x}_j] \in K \subset Z(G)$ and the space of all such sequences can be used to study $\mathrm{Hom}(\mathbb{Z}^n, G/K)$.

Definition 4. *Given a compact Lie group* G *and* $K \subset Z(G)$ *a closed subgroup define*

$$B_n(G, K) = \{(x_1, ..., x_n) \in G^n \mid [x_i, x_j] \in K \text{ for all } i, j\}.$$

The set $B_n(G, K)$ can be regarded as a topological space by naturally identifying it with a subspace of G^n. The following simple lemma describes the precise relationship between $B_n(G, K)$ and $\mathrm{Hom}(\mathbb{Z}^n, G/K)$.

Lemma 5. *Let* G *be a Lie group and* $K \subset Z(G)$ *a closed subgroup. Then the quotient map* $f : G \to G/K$ *induces a* G*-equivariant principal* K^n*-bundle*

$$\phi_n : B_n(G, K) \to \mathrm{Hom}(\mathbb{Z}^n, G/K).$$

In general K-almost commuting elements in G can be used to obtain a decomposition of the space $\mathrm{Hom}(\mathbb{Z}^n, G/K)$ into the union of (possibly empty) open and closed subspaces in the following way. Given $\underline{x} = (x_1, ..., x_n) \in B_n(G, K)$ consider the different commutators $d_{ij} = [x_i, x_j] \in K$ for $1 \leq i, j \leq n$. The elements d_{ij} are such that $d_{ii} = 1$ and $d_{ij} = d_{ji}^{-1}$, thus the matrix $D = (d_{ij})$ is an antisymmetric matrix with entries in $K \subset Z(G)$ that varies continuously with $\underline{x}$. Let $T(n, \pi_0(K))$ be the set of all $n \times n$ antisymmetric matrices $C = (c_{ij})$ with entries in $\pi_0(K)$. Given a matrix $C \in T(n, \pi_0(K))$ define

$$\mathcal{AC}_G(C) = \{(x_1, ..., x_n) \in G^n \mid \pi_0([x_i, x_j]) = c_{ij} \in \pi_0(K)\} \subset B_n(G, K),$$

and

$$\mathrm{Hom}(\mathbb{Z}^n, G/K)_C = \phi_n(\mathcal{AC}_G(C)) \subset \mathrm{Hom}(\mathbb{Z}^n, G/K).$$

Note that both $\mathcal{AC}_G(C)$ and $\mathrm{Hom}(\mathbb{Z}^n, G/K)_C$ are invariant under the conjugation action of G. Also these can be endowed with the natural subspace topology and in this case each $\mathrm{Hom}(\mathbb{Z}^n, G/K)_C$ is both open and closed in $\mathrm{Hom}(\mathbb{Z}^n, G/K)$ and thus a union of connected components. The restriction of ϕ_n defines a principal K^n-bundle

$$\mathcal{AC}_G(C) \to \mathrm{Hom}(\mathbb{Z}^n, G/K)_C$$

and there is a decomposition

$$\mathrm{Hom}(\mathbb{Z}^n, G/K) = \bigsqcup_{C \in T(n, \pi_0(K))} \mathrm{Hom}(\mathbb{Z}^n, G/K)_C. \tag{2}$$

In [5], Borel, Friedman and Morgan showed that the orbit space $\mathcal{M}_G(C) := \mathcal{AC}_G(C)/G$ is describable in terms of the geometry of G. Moreover, they obtained explicit descriptions for $n = 2$ and $n = 3$. In the next section, their work will be used to obtain an explicit description for $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})$ for every n. This sheds some light in the structure of the spaces of the form $\mathrm{Hom}(\mathbb{Z}^n, G)$ for a general compact Lie group G.

3. Commuting elements in $G_{m,p}$

The goal of this section is to prove Theorems 1 and 2 in the introduction. These are the main results of this article and are proved using decomposition (2).

To start, suppose that G is a compact connected Lie group. Let $\mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)}$ be the path-connected component of $\mathrm{Hom}(\mathbb{Z}^n, G)$ that contains $(1, ..., 1)$. By [1, Proposition 2.3], if every abelian subgroup of G is contained in a path-connected abelian subgroup, then the space $\mathrm{Hom}(\mathbb{Z}^n, G)$ is path-connected and thus agrees with $\mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)}$. In [3], the spaces of the form $\mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)}$ were studied. For example, the cohomology groups with rational coefficients of these spaces were computed. Some of the results proved in [3] are recalled next. The reader is referred to [3] for the proofs of these facts.

Fix $T \subset G$ a maximal torus in G. The conjugation action of G induces a G-equivariant map

$$\varphi_n : G \times T^n \to \mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)} \tag{3}$$

$$(g, t_1, ..., t_n) \mapsto (gt_1g^{-1}, ..., gt_ng^{-1}). \tag{4}$$

By [3, Lemma 4.2] it follows that every commuting n-tuple in $\mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)}$ lies in a maximal torus of G. Since any two maximal tori in G are conjugated this shows that the map φ_n is surjective. Note that $N(T)$ acts on $G \times T^n$ diagonally and that φ_n is invariant under this action. Therefore φ_n descends to a map

$$\bar{\varphi}_n : G/T \times_W T^n = G \times_{N(T)} T^n \to \mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)},$$

where W is the Weyl group associated to T. In fact $G \times_{N(T)} T^n$ is a nonsingular real algebraic variety and $\bar{\varphi}_n$ is a resolution of singularities for $\mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)}$ as it was pointed out in [3]. Thus in general $\mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)}$ is homeomorphic to the quotient of the compact manifold $G/T \times_W T^n$ where each fiber ${\bar{\varphi}_n}^{-1}(\underline{x})$ is collapsed to a point for $\underline{x} \in \mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)}$. Moreover, modulo the conjugation action of G, $\bar{\varphi}_n$ induces a homeomorphism

$$T^n/W \overset{\cong}{\to} \mathrm{Rep}(\mathbb{Z}^n, G)_{(1,...,1)},$$

with W acting diagonally on T^n. In addition, by [3, Theorem 4.3] given a field $\mathbb{F}$ of characteristic relatively prime to $|W|$, the map $\bar{\varphi}_n$ induces an isomorphism

$$H^*(\mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)}; \mathbb{F}) \cong H^*(G/T \times T^n; \mathbb{F})^W. \tag{5}$$

For the case of $G = G_{m,p}$, a maximal torus T is homeomorphic to $(\mathbb{S}^1)^{m(p-1)}$ and $W = (\Sigma_p)^m$. Moreover, if C_1 is the trivial matrix whose entries are all 1 then it follows that $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})_{(1,...,1)} = \mathrm{Hom}(\mathbb{Z}^n, G_{m,p})_{C_1}$ is a quotient of

$$(G_{m,p}/(\mathbb{S}^1)^{m(p-1)}) \times_{(\Sigma_p)^m} (\mathbb{S}^1)^{m(p-1)n}, \tag{6}$$

also

$$\mathrm{Rep}(\mathbb{Z}^n, G_{m,p})_{(1,...,1)} \cong (\mathbb{S}^1)^{m(p-1)n}/(\Sigma_p)^m \tag{7}$$

and

$$H^*(\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})_{(1,...,1)}; \mathbb{F}) \cong H^*((G_{m,p}/(\mathbb{S}^1)^{m(p-1)}) \times (\mathbb{S}^1)^{m(p-1)n}; \mathbb{F})^{(\Sigma_p)^m}. \tag{8}$$

for every field $\mathbb{F}$ with characteristic not dividing $p!$.

Next the spaces of the form $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})_C$ for $C \neq C_1$ are studied. The following lemma, which can be proved directly or using [5, Proposition 4.1.1], is used to handle this case.

Lemma 6. *Let $c \in Z(SU(p)) - \{1\}$. Then there is a pair (x_o, y_o) of elements in $SU(p)$ with $[x_o, y_o] = c$. Moreover, the pair (x_o, y_o) is unique up to conjugation and if (x, y) is any such pair then $Z_{SU(p)}(x, y) = Z(SU(p))$.*

The following notation will be used. Given an element $\underline{c} \in \Delta(\mathbb{Z}/p)$, $C(\underline{c})$ denotes the 2×2 antisymmetric matrix with entries in $\Delta(\mathbb{Z}/p)$ defined by $\underline{c}_{11} = \underline{c}_{22} = \underline{1} \in \Delta(\mathbb{Z}/p)$ and $\underline{c}_{12} = \underline{c}_{21}^{-1} = \underline{c}$. Theorem 1 will be proved by considering first the case $n = 2$.

Proposition 7. *The space $\mathrm{Hom}(\mathbb{Z}^2, G_{m,p})$ has p path–connected components. One of these components is $\mathrm{Hom}(\mathbb{Z}^2, G_{m,p})_{(1,...,1)}$ and the rest of the components are all homeomorphic to $SU(p)^m/((\mathbb{Z}/p)^{m-1} \times E_p)$.*

Proof: This proposition will be proved by studying the different spaces $\mathcal{AC}_{SU(p)^m}(C)$, where C is a general matrix in $T(2, \Delta(\mathbb{Z}/p))$. Such a matrix is of the form $C = C(\underline{c})$ for some $\underline{c} \in \Delta(\mathbb{Z}/p)$. When $\underline{c} = \underline{1}$ the space $\mathcal{AC}_{SU(p)^m}(C(\underline{1}))$ equals $\mathrm{Hom}(\mathbb{Z}^2, SU(p)^m)$ which is path–connected. Thus suppose that $\underline{c} \neq \underline{1}$. Since $\underline{c} \in \Delta(\mathbb{Z}/p)$, it is of the form $\underline{c} = (c, ..., c)$ for $c \in \mathbb{Z}/p = Z(SU(p))$ with $c \neq 1$. Fix a pair of elements

x_o, y_o in $SU(p)$ with $[x_o, y_o] = c$. By Lemma 6 the group $SU(p)^m$ acts transitively by conjugation on each $\mathcal{AC}_{SU(p)^m}(C(\underline{c}))$, thus there is a continuous surjective map

$$SU(p)^m \to \mathcal{AC}_{SU(p)^m}(C(\underline{c}))$$
$$(g_1, ..., g_m) \mapsto (\underline{x}, \underline{y})$$

where

$$\underline{x} = (g_1 x_o g_1^{-1}, ..., g_m x_o g_m^{-1}) \text{ and } \underline{y} = (g_1 y_o g_1^{-1}, ..., g_m y_o g_m^{-1}).$$

In particular, $\mathcal{AC}_{SU(p)^m}(C(\underline{c}))$ is path–connected and

$$\mathcal{AC}_{SU(p)^m}(C(\underline{c})) \cong SU(p)^m / SU(p)^m_{(\underline{x}_o, \underline{y}_o)},$$

where $\underline{x}_o = (x_o, ..., x_o)$, $\underline{y}_o = (y_o, ..., y_o)$ and $SU(p)^m_{(\underline{x}_o, \underline{y}_o)}$ is the isotropy subgroup of $SU(p)^m$ at $(\underline{x}_o, \underline{y}_o)$. Note that $Z_{SU(p)}(x_o, y_o) = Z(SU(p))$ by Lemma 6, hence

$$SU(p)^m_{(\underline{x}_o, \underline{y}_o)} = Z(SU(p)^m) = \langle c \rangle^m = (\mathbb{Z}/p)^m$$

and therefore

$$\mathcal{AC}_{SU(p)^m}(C(\underline{c})) \cong SU(p)^m / \langle c \rangle^m . \tag{9}$$

On the other hand, $(\Delta(\mathbb{Z}/p))^2$ acts on $\mathcal{AC}_{SU(p)^m}(C(\underline{c}))$ by left componentwise multiplication. This action gives rise to a covering space sequence

$$(\Delta(\mathbb{Z}/p))^2 \to \mathcal{AC}_{SU(p)^m}(C(\underline{c})) \to \mathrm{Hom}(\mathbb{Z}^2, G_{m,p})_{C(\underline{c})}. \tag{10}$$

In particular $\mathrm{Hom}(\mathbb{Z}^2, G_{m,p})_{C(\underline{c})}$ is path–connected and

$$\mathrm{Hom}(\mathbb{Z}^2, G_{m,p})_{C(\underline{c})} \cong \mathcal{AC}_{SU(p)^m}(C(\underline{c}))/(\Delta(\mathbb{Z}/p))^2.$$

Notice that under the identification (9), this action of $(\Delta(\mathbb{Z}/p))^2$ corresponds to

$$(\Delta(\mathbb{Z}/p))^2 \times SU(p)^m / \langle c \rangle^m \to SU(p)^m / \langle c \rangle^m$$
$$(\underline{c}^s, \underline{c}^r), [(g_1, ..., g_m)] \mapsto [(g_1 x_o^r y_o^{-s}, ..., g_m x_o^r y_o^{-s})].$$

This is true because

$$(x_o^r y_o^{-s}) x_o (x_o^r y_o^{-s})^{-1} = c^s x_o \text{ and } (x_o^r y_o^{-s}) y_o (x_o^r y_o^{-s})^{-1} = c^r y_o.$$

It follows then that $\mathrm{Hom}(\mathbb{Z}^2, G_{m,p})_{C(\underline{c})} \cong SU(p)^m / K_p$, where $K_p \subset SU(p)^m$ is the subgroup generated by $Z(SU(p)^m)$, $\underline{x}_o = (x_o, ..., x_o)$ and $\underline{y}_o = (y_o, ..., y_o)$. By [5, Proposition 4.1.1] the subgroup generated by x_o and y_o in $SU(p)/\langle c \rangle$ is isomorphic to $(\mathbb{Z}/p)^2$ and by [5, Corollary 4.1.2] x_o and y_o have order 4 and $x_o^2 = y_o^2 = c$ if $p = 2$ and order p if $p > 2$. Thus, when $p = 2$ the subgroup E_2 of $SU(2)$ generated by c, x_o and y_o has the presentation

$$E_2 := \left\{ x, y \mid x^4 = y^4 = 1, x^2 = y^2, yxy^{-1} = x \right\}$$

and thus $E_2 = Q_8$. When $p > 2$, the subgroup E_p of $SU(p)$ generated by c, x_o and y_o has the presentation

$$E_p = \{x, y, c \mid x^p = y^p = c^p = 1, xc = cx, yc = cy, xy = cyx\}$$

and this is easily seen to be the extraspecial p–group $Syl_p(SL_3(\mathbb{F}_p))$. The group K_p fits into a short exact sequence

$$1 \to (\mathbb{Z}/p)^{m-1} \to K_p \to \left\langle \underline{c}, \underline{x}_o, \underline{y}_o \right\rangle \to 1,$$

where the map $(\mathbb{Z}/p)^{m-1} \to K_p$ is as follows. Let $u_1, ..., u_{m-1}$ be elements in the $\mathbb{F}_p$ vector space $(\mathbb{Z}/p)^m$ such that $u_1, ..., u_{m-1}, \underline{c}$ forms a basis. Then the i-th generator of $(\mathbb{Z}/p)^{m-1}$ is sent to u_i for $1 \leq i \leq m-1$. The previous short exact sequences splits, $\left\langle \underline{c}, \underline{x}_o, \underline{y}_o \right\rangle \cong E_p$ and therefore $K_p \cong (\mathbb{Z}/p)^{m-1} \times E_p$. To finish the proposition, note there are precisely $(p-1)$ non-trivial elements $\underline{c} \in \Delta(\mathbb{Z}/p)$. □

From the previous proposition it is deduced that $N(2, m, p) = p$. Moreover, from the proof it follows that $G_{m,p}$ acts transitively by conjugation on each component that is homeomorphic to $SU(p)^m/((\mathbb{Z}/p)^{m-1} \times E_p)$.

Lemma 8. *Suppose that $\underline{x} = (x_1, ..., x_m)$ and $\underline{y} = (y_1, ..., y_m)$ are elements in $SU(p)^m$ that almost commute with $\underline{c} := [\underline{x}, \underline{y}] = (c, ..., c) \in \Delta(\mathbb{Z}/p)$ for $c \neq 1$. Take $\underline{z} \in SU(p)^m$ with $[\underline{x}, \underline{z}], [\underline{y}, \underline{z}] \in \Delta(\mathbb{Z}/p)$. Write $[\underline{x}, \underline{z}] = \underline{c}^b$ and $[\underline{y}, \underline{z}] = \underline{c}^a$ for integers $0 \leq a, b < p$. Then there is an element $\underline{w} = (w_1, ..., w_m) \in Z(SU(p)^m)$ such that $\underline{z} = \underline{w}\underline{x}^{-a}\underline{y}^b$; that is, $z_i = w_i x_i^{-a} y_i^b$ for all i.*

Proof: It is enough to prove the lemma for $m = 1$. Fix x, y and z in $SU(p)$ such that there exists $c \in Z(SU(p)) - \{1\}$ with $d_{1,2} := [x, y] = c$, $d_{1,3} := [x, z] = c^b$ and $d_{2,3} := [y, z] = c^a$ for integers $0 \leq a, b < p$. Then the triple (x, y, z) is an almost commuting triple in $\mathcal{AC}_{SU(p)}(D)$, where D is the antisymmetric matrix with entries $d_{i,j}$. Consider the map

$$\begin{aligned} \psi : \mathcal{AC}_{SU(p)}(C) &\to \mathcal{AC}_{SU(p)}(D) \\ (x_1, x_2, x_3) &\mapsto (x_1, x_2, x_1^{-a} x_2^b x_3), \end{aligned}$$

where C is the antisymmetric matrix with coefficients in $Z(SU(p))$ and $c_{1,2} = c_{2,1}^{-1} = c$ and $c_{i,j} = 1$ else. It is straight–forward to check that ψ is a well defined homeomorphism that is equivariant under the conjugation action of $SU(p)$. Let (x', y', z') be any element in $\mathcal{AC}_{SU(p)}(C)$. This means that $[x', y'] = c \neq 1$ and z' commutes with both x' and y'. Thus $z' \in Z_{SU(p)}(x', y') = Z(SU(p))$ by Lemma 6. On the other hand, since $[x, y] = c$ it follows by Lemma 6 that the pair (x', y') is conjugate to (x, y). This shows that any element in $\mathcal{AC}_{SU(p)}(C)$ is of the form (gxg^{-1}, gyg^{-1}, w) for some $g \in SU(p)$ and $w \in Z(SU(p))$. In particular, since ψ is surjective there are $g \in SU(p)$ and $w \in Z(SU(p))$ such that

$$(x, y, z) = \psi(gxg^{-1}, gyg^{-1}, w);$$

that is,

$$(x, y, z) = (gxg^{-1}, gyg^{-1}, g(wx^{-a}y^b)g^{-1}).$$

This means that $gx = xg$ and $gy = yg$, hence $g \in Z_{SU(p)}(x, y) = Z(SU(p))$ and therefore

$$z = wx^{-a}y^b.$$

□

The next step is the proof of Theorem 1 in the introduction.

Theorem 1. *For $n \geq 1$ and p a prime number, the space* $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})$ *has*

$$N(n,m,p) = \frac{p^{(m-1)(n-2)}(p^n-1)(p^{n-1}-1)}{p^2-1} + 1$$

path–connected components. The path-connected component containing $(1,...,1)$ *is a quotient of, and has the same rational cohomology as,*

$$(G_{m,p}/(\mathbb{S}^1)^{m(p-1)}) \times_{(\Sigma_p)^m} (\mathbb{S}^1)^{m(p-1)n},$$

whereas all the other path–connected components are homeomorphic to

$$(SU(p))^m/((\mathbb{Z}/p)^{m-1} \times E_p),$$

where $E_p \subset SU(p)$ is the quaternion group Q_8 of order eight when $p = 2$ and the extraspecial p–group of order p^3 and exponent p when $p > 2$.

Proof: Fix p a prime number. The proof of the theorem goes by induction on n. For $n = 1$ the theorem is trivial and for $n = 2$ the theorem follows by the Proposition 7. Assume then that $n \geq 3$. To determine the value of each $N(n,m,p)$ it will be shown that the different $N(n,m,p)$'s satisfy the recurrence equation

$$N(n,m,p) = p^{m-1}N(n-1,m,p) + p^{m(n-2)+n-1} - p^{m(n-2)} - p^{m-1} + 1.$$

Once this proved, by induction it follows that

$$N(n,m,p) = \frac{p^{(m-1)(n-2)}(p^n-1)(p^{n-1}-1)}{p^2-1} + 1.$$

By (2) the space $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})$ is a disjoint union of the different $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})_C$, where C runs through the elements in $T(n, \Delta(\mathbb{Z}.p))$. The different possibilities for elements $C \in T(n, \Delta(\mathbb{Z}/p))$ are considered next.

• **Case 1**. Suppose that $C = C_1 \in T(n, \Delta(\mathbb{Z}/p))$ is the trivial matrix whose entries are all equal to 1. In this case the space $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})_{C_1}$ is path–connected and as described in (6) and (8).

• **Case 2**. Suppose that $C \in T(n, \Delta(\mathbb{Z}/p)) - \{C_1\}$ is such that $\underline{c}_{1,i} = 1$ for all i. Because C is not trivial there exist $2 \leq i, j \leq n$ such that $\underline{c}_{i,j} \neq 1$. Take $(\underline{x}_1, ..., \underline{x}_n) \in \mathcal{AC}_{SU(p)^m}(C)$. Since $\underline{c}_{1,i} = 1$, it follows that $\underline{x}_1$ commutes with $\underline{x}_i$ for all i. Also, $[\underline{x}_i, \underline{x}_j] \in \Delta(\mathbb{Z}/p) - \{1\}$ and thus $\underline{x}_1 \in Z_{SU(p)^m}(\underline{x}_i, \underline{x}_j) = Z(SU(p)^m)$ by Lemma 6. Therefore $(\underline{x}_1, ..., \underline{x}_n) \in Z(SU(p)^m) \times \mathcal{AC}_{SU(p)}(\tilde{C})$, where $\tilde{C}$ is the $(n-1) \times (n-1)$ matrix obtained from C by deleting the first row and column from C. In this case

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})_C &= (Z(SU(p)^m) \times \mathcal{AC}_{SU(p)^m}(\tilde{C}))/(\Delta(\mathbb{Z}/p))^n \\ &\cong (\mathbb{Z}/p)^m/(\Delta(\mathbb{Z}/p)) \times \mathrm{Hom}(\mathbb{Z}^{n-1}, G_{m,p})_{\tilde{C}} \\ &\cong (\mathbb{Z}/p)^{m-1} \times \mathrm{Hom}(\mathbb{Z}^{n-1}, G_{m,p})_{\tilde{C}}.\end{aligned}$$

By induction each path–connected component of $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})_C$ is homeomorphic to

$$SU(p)^m/((\mathbb{Z}/p)^{m-1} \times E_p)$$

with $G_{m,p}$ acting transitively by conjugation. In addition, each matrix C of the type considered in this case determines and is uniquely determined by the corresponding $\tilde{C}$ which is non trivial. It follows that there are $p^{m-1}(N(n-1,m,p)-1)$ path–connected components associated to this case.

• **Case 3**. Suppose that $C \in T(n, \Delta(\mathbb{Z}/p))$ is such that $\underline{c}_{1i} \neq 1$ for some i. Then $2 \leq i \leq n$ as $\underline{c}_{11} = \underline{1}$. Let i be the smallest i with $\underline{c}_{1i} \neq 1$, let $\underline{c} = \underline{c}_{1i} \in \Delta(\mathbb{Z}/p)$ and take $(\underline{x}_1, ..., \underline{x}_n) \in \mathcal{AC}_{SU(p)^m}(C)$. For each $2 \leq k \leq n$ with $k \neq i$ consider the triple $(\underline{x}_1, \underline{x}_i, \underline{x}_k)$. This is an almost commuting triple with $[\underline{x}_1, \underline{x}_i] = \underline{c} \neq 1$. By Lemma 8, if $\underline{c}_{1,k} = [\underline{x}_1, \underline{x}_k] = \underline{c}^{b_k}$ and $\underline{c}_{i,k} = [\underline{x}_i, \underline{x}_k] = \underline{c}^{a_k}$ for integers $0 \leq a_k, b_k < p$, then there exist $\underline{w}_k \in Z(SU(p)^m)$ such that $\underline{x}_k = \underline{w}_k \underline{x}_1^{-a_k} \underline{x}_i^{b_k}$ for all k. Note that the integers a_k and b_k are uniquely determined by the condition $0 \leq a_k, b_k < p$ and these are in turn uniquely determined by $\underline{c}_{1k}, \underline{c}_{ik}$. It follows that the n-tuple $(\underline{x}_1, ..., \underline{x}_n)$ is uniquely determined by $(\underline{x}_1, \underline{x}_i)$, $\underline{c}_{1,k}, \underline{c}_{i,k} \in \Delta(\mathbb{Z}/p)$ and $\underline{w}_k \in Z(SU(p)^m)$ for $k \neq 1, i$. Moreover, if as before $C(\underline{c})$ is the 2×2 matrix

$$C(\underline{c}) = \begin{bmatrix} 1 & \underline{c} \\ \underline{c}^{-1} & 1 \end{bmatrix},$$

then the map

$$\psi : \mathcal{AC}_{SU(p)^m}(C(\underline{c})) \times (Z(SU(p)^m))^{n-2} \to \mathcal{AC}_{SU(p)^m}(C)$$
$$((\underline{x}_1, \underline{x}_i), (\underline{w}_2, ..., \underline{w}_{i-1}, \underline{w}_{i+1}, ..., \underline{w}_n)) \mapsto (\underline{y}_1, ..., \underline{y}_n),$$

is a homeomorphism where

$$\underline{y}_k = \begin{cases} \underline{x}_1 & \text{if } k = 1, \\ \underline{x}_i & \text{if } k = i, \\ \underline{w}_k \underline{x}_1^{-a_k} \underline{x}_i^{b_k} & \text{if } k \neq 1, i. \end{cases}$$

The map ψ is $SU(p)^m$-equivariant, with $SU(p)^m$ acting by conjugation. By passing to the quotient of the respective $(\Delta(\mathbb{Z}/p))^n$–actions, it follows that ψ induces a homeomorphism

$$\mathrm{Hom}(\mathbb{Z}^2, G_{m,p})_{C(\underline{c})} \times (\mathbb{Z}/p)^{(m-1)(n-2)} \to \mathrm{Hom}(\mathbb{Z}^n, G_{m,p})_C.$$

By the case $n = 2$, each path–connected component of $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})_C$ is of the desired type and there are $p^{(m-1)(n-2)}$ such components associated to C. It also follows that $G_{m,p}$ acts transitively on each of these components. Moreover, C is uniquely determined by $\underline{c} = \underline{c}_{1i} \neq 1$, $\underline{c}_{1k}$ for $i+1 \leq k \leq n$ and $\underline{c}_{ik}$ for $2 \leq k \leq n$ and $k \neq i$. Thus there are in total $p^{(m-1)(n-2)}(p-1)p^{2n-i-2}$ different components associated to such C with $\underline{c}_{1,i} \neq 1$. Letting $2 \leq i \leq n$ vary, a total number of

$$\sum_{i=2}^{n} p^{(m-1)(n-2)}(p-1)p^{2n-i-2} = p^{m(n-2)}(p^{n-1} - 1)$$

path–connected components is obtained for this case.

By adding the contributions from case 1, case 2 and case 3 the recurrence equation

$$\begin{aligned} N(n,m,p) &= 1 + p^{m-1}(N(n-1,m,p)-1) + p^{m(n-2)}(p^{n-1}-1) \\ &= p^{m-1}N(n-1,m,p) + p^{m(n-2)+n-1} - p^{m(n-2)} - p^{m-1} + 1. \end{aligned}$$

is obtained as claimed. □

As mentioned before $G_{m,p}$ acts transitively on the components of $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})$ that are homeomorphic to $SU(p)^m/(\mathbb{Z}/p^{m-1} \times E_p)$. This shows that these path-connected components represent isolated points in the moduli space $\mathrm{Rep}(\mathbb{Z}^n, G_{m,p})$. On the other hand, by (7) there is a homeomorphism

$$\mathrm{Rep}(\mathbb{Z}^n, G_{m,p})_{(1,\dots,1)} \cong (\mathbb{S}^1)^{m(p-1)n}/(\Sigma_p)^m.$$

As a corollary of this the following theorem is obtained.

Theorem 2. *Let p be a prime number and $m \geq 1$. Then $\mathrm{Rep}(\mathbb{Z}^n, G_{m,p})$ has*

$$N(n,m,p) = \frac{p^{(m-1)(n-2)}(p^n-1)(p^{n-1}-1)}{p^2-1} + 1$$

path–connected components and

$$\mathrm{Rep}(\mathbb{Z}^n, G_{m,p}) \cong ((\mathbb{S}^1)^{(p-1)mn}/(\Sigma_p)^m) \sqcup X_{n,m,p}.$$

where $X_{n,m,p}$ is a finite set with $N(n,m,p)-1$ points.

The component of the identity can be described more explicitly as follows. Σ_p acts on $(\mathbb{S}^1)^p$ as the Weyl group of a maximal torus in $SU(p)$. Then the product $(\Sigma_p)^m$ acts on the product $(\mathbb{S}^1)^{(p-1)m}$, and therefore diagonally on the product $(\mathbb{S}^1)^{(p-1)mn}$. For example, if $p = 2$, the action of $(\Sigma_2)^m$ on $(\mathbb{S}^1)^m$ is simply given as a product of the complex conjugation action, and this is extended to a diagonal action on $((\mathbb{S}^1)^m)^n$.

In Theorem 1 if p is not longer assumed to be a prime number then the situation is more complicated. For example when $n = 2$, the conjugation action of $SU(r)$ on $\mathcal{AC}_{SU(r)}(C(c))$ is not longer transitive, unless c is a generator of the the cyclic group $\mathbb{Z}/r$. Because of this, the space $\mathrm{Hom}(\mathbb{Z}^n, SU(r))$ has in general more path–connected components that have orbifold singularities. In particular, for $n = 2$ there is the following proposition that can be proved in the same way as Lemma 7.

Proposition 9. *The space $\mathrm{Hom}(\mathbb{Z}^2, PU(r))$ has r path–connected components. Of these $\varphi(r)$ are homeomorphic to $PU(r)/(\mathbb{Z}/r)^2$.*

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