

From fibered symmetric bimonoidal categories to symmetric spectra

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Abstract

In here we define the concept of fibered symmetric bimonoidal categories. These are roughly speaking fibered categories $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ whose fibers are symmetric monoidal categories parametrized by \mathcal{C} and such that both \mathcal{D} and \mathcal{C} have a further structure of a symmetric monoidal category that satisfy certain coherences that we describe. Our goal is to show that we can correspond to a fibered symmetric bimonoidal category an E_∞ -ring spectrum in a functorial way.

Key words: Elliptic cohomology, K-theory, permutative category, symmetric spectra, E_∞ ring spectra.

1 Introduction

The goal of this paper is to study in general what we denote by fibered symmetric bimonoidal categories. These are fibered categories $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$, such that \mathcal{D} and \mathcal{C} are topological categories and such that each fiber \mathcal{D}_c has the structure of a symmetric monoidal category $(\mathcal{D}_c, \oplus_c, 0_c)$. In addition, both \mathcal{D} and \mathcal{C} are symmetric monoidal categories, $(\mathcal{D}, \otimes, 1)$ and $(\mathcal{C}, \otimes, 1)$, and the functor Λ is a continuous symmetric monoidal functor. The operations \otimes and \oplus_c , are compatible in the sense that they satisfy some coherences. These coherences are similar to those satisfied by a symmetric bimonoidal category plus some new coherences that we require. (For the precise definition see Definition 1 below). We show that given a fibered symmetric bimonoidal category, we can correspond an E_∞ -ring spectrum in a functorial way.

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Our motivation to study fibered symmetric bimonoidal categories comes from an attempt to produce a geometric based model for elliptic cohomology. In (9), Hu and Kriz proposed a construction of an elliptic cohomology type spectrum based on a concept that they referred to as elliptic bundles. These are defined to be stringy bundles on a fixed elliptic curve that are invariant under translations. Roughly speaking, a stringy bundle is a holomorphic analogue of Segal’s elliptic objects (see (9) for definitions). The category of elliptic bundles and isomorphisms between them is a symmetric monoidal category under a product \otimes . One difficulty in the construction of (9) is, that as defined, there is no obvious way to add elliptic bundles. Hu and Kriz get around this inconvenience by using some machinery from homotopy theory. In (5), Gómez, Hu and Kriz propose a geometric model of elliptic cohomology using a modified version of stringy bundles over \mathbb{C} with compact supports. This produces an example of a fibered symmetric bimonoidal category which is our motivating example. More precisely, our main motivation for studying fibered symmetric bimonoidal categories is the following construction. Let \mathcal{C} be the SPCMC of worldsheets over the stack of finite dimensional complex manifolds. Given a Riemann surface X we can associate a SPCMC \mathcal{C}_X (see (5) for the definition). We have the topological category \mathbf{A} of conformal field theories (CFT’s) on \mathcal{C} which is a fibered symmetric bimonoidal category over the discrete category \mathcal{MF} of modular functors over \mathcal{C} . In addition, we have the topological category \mathbf{B} of CFT’s $\mathcal{C}_{\mathbb{C}}$ over modular functors pullback from \mathcal{C} . This is also fibered symmetric bimonoidal category. Using the machinery that we describe here we can correspond to this data, two E_{∞} -ring spectra $\mathcal{E}_{\mathbf{A}}$ and $\mathcal{E}_{\mathbf{B}}$, together with a map of spectra $\phi : \mathcal{E}_{\mathbf{A}} \rightarrow \mathcal{E}_{\mathbf{B}}$. The fiber of this map is our proposed elliptic cohomology type spectrum. (See (5) for more details).

We begin our study of fibered symmetric bimonoidal categories by first studying the discrete case; that is, we first study the case where all the structure insight is discrete. By applying a streefication process we show that in general a discrete fibered symmetric bimonoidal category can be replaced by an equivalent fibered category $\Lambda : \mathcal{D}^s \rightarrow \mathcal{C}^s$ such that $(\mathcal{D}^s, \otimes, 1)$ and $(\mathcal{C}^s, \otimes, 1)$ are permutative categories and $\Lambda^s : \mathcal{D}^s \rightarrow \mathcal{C}^s$ is a strict map. In addition for every object c of \mathcal{C}^s the fibers \mathcal{D}_c^s have the structure of a permutative category $(\mathcal{D}_c^s, \oplus_c, 0_c)$. We also have distributivity maps satisfying coherences similar to those satisfied by a bipermutative category. Thus $\Lambda^s : \mathcal{D}^s \rightarrow \mathcal{C}^s$ is what we call a discrete fibered bipermutative category. (See Definition 6 below).

In general, having a fibered category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ is equivalent to having a contravariant lax 2-functor $\mathcal{C}^{op} \rightarrow \mathbf{Cat}$ and the latter can be replaced by an equivalent functor $\mathcal{C}^{op} \rightarrow \mathbf{Cat}$. This is shown for example in (16, Sections 3.1.2 and 3.1.3). We use this idea as our starting point. Thus we can correspond to a fibered bipermutative category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$, a functor $\mathcal{C}^{op} \rightarrow \mathbb{P}$, where \mathbb{P} is the category of small permutative categories. However, in order to capture the multiplicative structure present we need to enlarge the category \mathcal{C}^{op} . Thus

given a fibered bipermutative category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ we consider the permutative category \mathcal{A} that is defined as the wreath product category $\mathbf{Inj} f(\mathcal{C}^{op})^*$, for a functor

$$(\mathcal{C}^{op})^* : \mathbf{Inj} \rightarrow \mathbb{P}.$$

We show that given a fibered bipermutative category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$, we can correspond a functor $\Psi : \mathcal{A} \rightarrow \mathbb{P}$. This functor preserves the multiplicative structure in the sense that for objects u and v of \mathcal{A} , we have a functor

$$\otimes_{u,v} : \Psi(u) \times \Psi(v) \rightarrow \Psi(u \odot v)$$

that satisfies certain conditions. (These are conditions **(c.1)** – **(c.14)** of Theorem 25).

The functor $\Psi : \mathcal{A} \rightarrow \mathbb{P}$ induced by a fibered bipermutative category motivates the study in general of the category of functors $\mathcal{E} \rightarrow \mathbb{P}$, where $(\mathcal{E}, \otimes, 1)$ is a general permutative category. We define a multicategory structure on this category. This is a special case of a more general construction; that is, if \mathcal{M} is a multicategory, then we show that we can find a multicategory $\mathcal{M}^\mathcal{E}$ whose objects are the functors $\mathcal{E} \rightarrow \mathcal{M}$. In addition, we show that for the functor $\Psi : \mathcal{A} \rightarrow \mathbb{P}$ induced by a fibered bipermutative category the additional multiplicative structure given by the permutative structure on \mathcal{D} and the coherences gives rise to an enriched multifunctor over **Cat**

$$T_1 : E\Sigma_* \rightarrow \mathbb{P}^{\mathcal{A}}.$$

Here $E\Sigma_*$ is the category valued operad, whose value at $k \geq 0$, is the translation category $E\Sigma_k$ that has as object set Σ_k and there is only one morphism between any two objects of $E\Sigma_k$.

We also show that given an enriched multifunctor $\mathcal{F} : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ between two multicategories \mathcal{M}_1 and \mathcal{M}_2 enriched over **Cat**, then composition with \mathcal{F} gives rise to a multifunctor $\mathcal{F}_* : \mathcal{M}_1^\mathcal{E} \rightarrow \mathcal{M}_2^\mathcal{E}$. In particular, if we take the multifunctor $K : \mathbb{P} \rightarrow \Sigma\mathcal{S}$ defined by Elemendorf and Mandell (4, Theorem 1.1), we get a multifunctor $K_* : \mathbb{P}^\mathcal{E} \rightarrow \Sigma\mathcal{S}^\mathcal{E}$. By composing the multifunctors T_1 and K_* in the case of $\mathcal{E} = \mathcal{A}$, we obtain a multifunctor $T : E\Sigma_* \rightarrow \Sigma\mathcal{S}^{\mathcal{A}}$. In addition, we show that in general, given a multicategory \mathcal{M} enriched over simplicial sets, then the positive model structure on $\Sigma\mathcal{S}$ can be lifted as to get a closed model category structure on the category of multifunctors $\mathcal{M} \rightarrow \Sigma\mathcal{S}^\mathcal{E}$. We use this model structure to show that a multifunctor $T : E\Sigma_* \rightarrow \Sigma\mathcal{S}^\mathcal{E}$ can be rectified as to obtain a multifunctor $T' : * \rightarrow \Sigma\mathcal{S}^\mathcal{E}$. It turns out that having such a multifunctor is equivalent as having a lax map $\mathcal{E} \rightarrow \Sigma\mathcal{S}$. Thus in the case of \mathcal{A} , we obtain a lax map $\vartheta : \mathcal{A} \rightarrow \Sigma\mathcal{S}$.

We have a canonical map $\mathcal{W} : \mathcal{A} \rightarrow \mathcal{C}^{op}$ that is a strict map. Using the model structure mentioned above, we show that this map induces, via a left adjoint in a Quillen adjunction, a lax map $\phi : \mathcal{C}^{op} \rightarrow \Sigma\mathcal{S}$. In a similar way, we show

that such a functor gives rise to a lax map $\phi' : (\mathcal{C}^{op})^{-1}\mathcal{C}^{op} \rightarrow \Sigma\mathcal{S}$. In particular, the image of the unit give us a strictly commutative symmetric ring spectrum. The outlined construction is functorial, thus we obtain a functor

$$\mathcal{Z} : \mathcal{FS} \rightarrow \Sigma\mathcal{CRS},$$

where \mathcal{FS} is the category of discrete fibered categories and $\Sigma\mathcal{CRS}$ is the category of strictly commutative symmetric ring spectra.

Finally, given a topological fibered category, by applying the singular functor, we can see it as a simplicial fibered category; that is, given a topological fibered category we correspond a functor

$$\Delta^{op} \rightarrow \mathcal{FS}$$

and by composing this functor with the functor \mathcal{Z} we obtain a simplicial commutative ring spectrum whose realization is the desired spectrum.

A big part of this work was inspired from the ideas of Elmendorf and Mandell in their beautiful work (4) and (3). Some propositions and theorems here are direct adaptations of their ideas to our settings. Also I would like to thank Igor Kriz and Po Hu for all their help and useful comments throughout this work.

2 From fibered symmetric bimonoidal categories to fibered bipermutative categories

In this section we define fibered symmetric bimonoidal categories and fibered bipermutative categories. Then we show, using a standard procedure in category theory, that every fibered symmetric bimonoidal category can be replaced by an equivalent fibered bipermutative category. The latter will be used as input for the machine that we construct and that produces an E_∞ -ring spectrum.

Definition 1 *A discrete fibered symmetric bimonoidal category is a fibered category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$, (see Definition 10), where $(\mathcal{D}, \otimes, 1, \gamma^\otimes)$ and $(\mathcal{C}, \otimes, 1, \gamma^\otimes)$ are two small symmetric monoidal categories and Λ is a symmetric monoidal functor. In addition, for each object c of \mathcal{C} , the fiber \mathcal{D}_c has the structure of a symmetric monoidal category $(\mathcal{D}_c, \oplus_c, 0_c, \gamma_c^\oplus)$. These structures are compatible in the sense that we have natural distributivity maps*

$$\begin{aligned} d^l &: (x \otimes y) \oplus (x' \otimes y) \rightarrow (x \oplus x') \otimes y, \\ d^r &: (x \otimes y) \oplus (x \otimes y') \rightarrow x \otimes (y \oplus y'), \end{aligned}$$

defined whenever x and x' , y and y' are in fibers \mathcal{D}_c , \mathcal{D}_d respectively. The morphisms d^l and d^r are morphisms in $\mathcal{D}_{\Lambda(x \otimes y)} = \mathcal{D}_{\Lambda((x \oplus x') \otimes y)}$. We also

have natural isomorphisms

$$\begin{aligned}\lambda'_c : 0_c \oplus x &\rightarrow x, & \rho'_c : x \oplus 0_c &\rightarrow x'_c \\ \lambda_c^* : 0_d \otimes x &\rightarrow 0_{d \otimes c}, & \rho_d^* : x \otimes 0_d &\rightarrow 0_{c \otimes d},\end{aligned}$$

for an object x in \mathcal{D}_c . We require similar coherences than those described by Laplaza for symmetric bimonoidal categories as described in (10) to hold whenever they make sense with the zero and null morphisms replaced by the above morphisms. In addition, we can add morphisms in \mathcal{D} over a same morphism in \mathcal{C} ; that is, given morphisms $\alpha : x \rightarrow y$ and $\beta : x' \rightarrow y'$ with $\Lambda(\alpha) = \Lambda(\beta) = f$, for $f : c \rightarrow d$, then we can find a morphism

$$\alpha \oplus \beta : x \oplus x' \rightarrow y \oplus y',$$

with $\Lambda(\alpha \oplus \beta) = f$. In the case that $f = id_c$, then $\alpha \oplus \beta = \alpha \oplus_c \beta$. We require the following diagrams to be commutative:

$$\begin{array}{ccc} x \oplus x' & \xrightarrow{\alpha \oplus \beta} & y \oplus y' \\ \gamma_c^\oplus \downarrow & & \downarrow \gamma_d^\oplus \\ x' \oplus x & \xrightarrow{\beta \oplus \alpha} & y' \oplus y \end{array}$$

$$\begin{array}{ccc} (x \oplus x') \oplus x'' & \xrightarrow{(\alpha \oplus \beta) \oplus \delta} & (y \oplus y') \oplus y'' \\ \cong \downarrow & & \downarrow \cong \\ x \oplus (x' \oplus x'') & \xrightarrow{\alpha \oplus (\beta \oplus \delta)} & y \oplus (y' \oplus y'') \end{array}$$

Also, given any morphism $f : c \rightarrow d$ in \mathcal{C} we can find a morphism $0_f : 0_c \rightarrow 0_d$ over f such that

$$\begin{array}{ccc} x \oplus 0_c & \xrightarrow{\alpha \oplus 0_f} & y \oplus 0_d \\ \cong \downarrow & & \downarrow \cong \\ x & \xrightarrow{\alpha} & y. \end{array}$$

The morphisms 0_f 's are defined in a functorial way; that is, given $f : c \rightarrow d$ and $g : d \rightarrow e$ morphisms in \mathcal{C} , then $0_{g \circ f} = 0_g \circ 0_f$ and if $f = id : c \rightarrow c$, then $0_{id_c} = id_{0_c}$.

Remark 2 The naturality statement for all the morphisms in the above definition means naturality whenever this makes sense, for example for the distributivity morphisms d^l and d^r , the naturality means that given $g : x \rightarrow x_1$ and $g' : x' \rightarrow x'_1$ morphisms over $f : c \rightarrow c'$ and a morphism $h : y \rightarrow y'$ over $f' : d \rightarrow d'$, then the following diagram is commutative

$$\begin{array}{ccc} (x \otimes y) \oplus (x' \otimes y) & \xrightarrow{d^l} & (x \oplus x') \otimes y \\ (g \otimes h) \oplus (g' \otimes h) \downarrow & & \downarrow (g \oplus g') \otimes h \\ (x_1 \otimes y_1) \oplus (x'_1 \otimes y_1) & \xrightarrow{d^l} & (x_1 \oplus x'_1) \otimes y_1 \end{array}$$

and similarly for d^r .

Remark 3 If \mathcal{C} is the trivial category with only one object and one morphism, then a discrete fibered symmetric bimonoidal category over \mathcal{C} is just a symmetric bimonoidal category.

In the case of symmetric bimonoidal categories, it is convenient to work in a more rigid scenario by requiring the operations in sight to be strictly associative and to have strict units. Similarly, we have a strict version of a fibered symmetric bimonoidal category. We call these fibered bipermutative categories. Before giving the precise definition, we recall the definition of a permutative category.

Definition 4 A permutative category consists of a category \mathcal{C} , a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ that is strictly associative, a strict unit 1 ; that is, $x \otimes 1 = 1 = 1 \otimes x$ for all objects x and a natural isomorphism $\gamma : x \otimes y \rightarrow y \otimes x$ such that the following diagrams are commutative

$$\begin{array}{ccc}
 x \otimes 1 & \xrightarrow[\cong]{\gamma} & 1 \otimes x \\
 \searrow = & & \swarrow = \\
 & x &
 \end{array}
 \qquad
 \begin{array}{ccc}
 x \otimes y & \xrightarrow{=} & x \otimes y \\
 \searrow \cong \gamma & & \swarrow \cong \gamma \\
 & y \otimes x &
 \end{array}$$

$$\begin{array}{ccc}
 x \otimes y \otimes z & \xrightarrow{\gamma} & z \otimes x \otimes y \\
 \searrow \text{id} \otimes \gamma & & \swarrow \gamma \otimes \text{id} \\
 & x \otimes z \otimes y &
 \end{array}$$

We have three different notions of morphisms between permutative categories that will be useful for us. Suppose that $(\mathcal{D}, \otimes, 1)$ and $(\mathcal{E}, \otimes, 1)$ are permutative categories, then:

- A functor $f : \mathcal{C} \rightarrow \mathcal{D}$ is said to be a strict map if $f(x \otimes y) = f(x) \otimes f(y)$, $f(1) = 1$ and

$$\begin{array}{ccc}
 f(x \otimes y) & \xrightarrow{=} & f(x) \otimes f(y) \\
 f(\gamma) \downarrow & & \downarrow \gamma \\
 f(y \otimes x) & \xrightarrow{=} & f(y) \otimes f(x)
 \end{array}$$

is a commutative diagram.

- A functor $f : \mathcal{C} \rightarrow \mathcal{D}$ is said to be a lax_{*} map if $f(1) = 1$ and there exists a natural transformation

$$\lambda = \lambda_f : f(x) \otimes f(y) \rightarrow f(x \otimes y)$$

such that $\lambda = \text{id}$ whenever $x = 1$ or $y = 1$ and the following diagrams are

commutative

$$\begin{array}{ccc} f(x) \otimes f(y) \otimes f(z) & \xrightarrow{\text{id} \otimes \lambda} & f(x) \otimes f(y \otimes z) \\ \lambda \otimes \text{id} \downarrow & & \downarrow \lambda \\ f(x \otimes y) \otimes f(z) & \xrightarrow{\lambda} & f(x \otimes y \otimes z), \end{array}$$

$$\begin{array}{ccc} f(x) \otimes f(y) & \xrightarrow{\lambda} & f(x \otimes y) \\ \gamma \downarrow & & \downarrow f(\gamma) \\ f(y) \otimes f(x) & \xrightarrow{\lambda} & f(y \otimes x). \end{array}$$

- A functor $f : \mathcal{C} \rightarrow \mathcal{D}$ is said to be a lax map, if we can find a map $\eta : 1 \rightarrow f(1)$ and a natural transformation

$$\lambda = \lambda_f : f(x) \otimes f(y) \rightarrow f(x \otimes y)$$

such that the following similar coherences are satisfied together with those coherences involving the unit.

From now on we will denote by \mathbb{P} the category whose objects are the small permutative categories and whose morphisms are the lax_{*} maps. This category \mathbb{P} will play a crucial roll throughout this work.

Definition 5 A *bipermutative category* is category that has two permutative category structures $(\mathcal{C}, \oplus, 0, \gamma^\oplus)$ and $(\mathcal{C}, \otimes, 1, \gamma^\otimes)$. These satisfy that $x \otimes 0 = 0 = 0 \otimes x$ and we have natural distributivity isomorphisms

$$\begin{aligned} d^l &: (x \otimes y) \oplus (x' \otimes y) \rightarrow (x \oplus x') \otimes y, \\ d^r &: (x \otimes y) \oplus (x \otimes y') \rightarrow x \otimes (y \oplus y'), \end{aligned}$$

that satisfy the following coherences:

$$\begin{array}{ccc} (x \otimes y) \oplus (x' \otimes y) \oplus (x'' \otimes y) & \xrightarrow{d^\oplus \oplus \text{id}} & ((x \oplus x') \otimes y) \oplus (x'' \otimes y) \\ \text{id} \oplus d^l \downarrow & & \downarrow d^l \\ (x \otimes y) \oplus ((x' \oplus x'') \otimes y) & \xrightarrow{d^l} & (x \oplus x' \oplus x'') \otimes y, \end{array} \quad (\mathbf{a.1})$$

$$\begin{array}{ccc} (x \otimes y) \oplus (x' \otimes y) & \xrightarrow{d^l} & (x \oplus x') \otimes y \\ \gamma^\oplus \downarrow & & \downarrow \gamma^\oplus \otimes \text{id} \\ (x' \otimes y) \oplus (x \otimes y) & \xrightarrow{d^l} & (x' \oplus x) \otimes y, \end{array} \quad (\mathbf{a.2})$$

$$\begin{array}{ccc} (x \otimes y \otimes z) \oplus (x' \otimes y \otimes z) & \xrightarrow{d^l} & (x \oplus x') \otimes y \otimes z \\ d^l \downarrow & \nearrow d^\otimes \otimes \text{id} & \\ ((x \otimes y) \oplus (x' \otimes y)) \otimes z & & \end{array} \quad (\mathbf{a.3})$$

are commutative diagrams. Also we have similar commutative diagrams with d^l replaced by d^r . Also the following diagrams are commutative:

$$\begin{array}{ccc}
& (x \otimes (y \oplus y')) \oplus (x' \otimes (y \oplus y')) & \text{(a.4)} \\
& \xrightarrow{d^r \oplus d^r} & \\
(x \otimes y) \oplus (x \otimes y') \oplus (x' \otimes y) \oplus (x' \otimes y') & & \\
\downarrow \text{id} \oplus \gamma^{\oplus} \oplus \text{id} & & \\
(x \otimes y) \oplus (x' \otimes y) \oplus (x \otimes y') \oplus (x' \otimes y') & & \\
& \searrow^{d^l \oplus d^l} & \\
& ((x \oplus x') \otimes y) \oplus ((x \oplus x') \otimes y') & \\
& \nearrow^{d^r} & \\
& (x \oplus x') \otimes (y \oplus y'), & \\
& \nwarrow^{d^l} & \\
& (x \otimes (y \oplus y')) \oplus (x' \otimes (y \oplus y')) &
\end{array}$$

$$\begin{array}{ccc}
(x \otimes y) \oplus (x' \otimes y) & \xrightarrow{d^l} & (x \oplus x') \otimes y & \text{(a.5)} \\
\downarrow \gamma^{\otimes} \oplus \gamma^{\otimes} & & \downarrow \gamma^{\otimes} & \\
(y \otimes x) \oplus (y \otimes x') & \xrightarrow{d^r} & y \otimes (x \oplus x'). &
\end{array}$$

Definition 6 A fibered category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ is said to be a fibered bipermutative category if it satisfies the following properties: $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ is a fibered category such that $(\mathcal{D}, \otimes, 1)$ and $(\mathcal{C}, \otimes, 1)$ are small permutative categories and $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ is a strict map. In addition, we require that the fibers of Λ have an additional structure of a permutative category $(\mathcal{D}_c, \oplus_c, 0_c, \gamma_c^{\oplus})$. We assume that $x \otimes 0_d = 0_{c \otimes d}$ and $0_c \otimes y = 0_{c \otimes d}$ for all objects x and y of \mathcal{D}_c and \mathcal{D}_d respectively. We also require the operations \otimes and \oplus_c to be compatible in the sense that we can find natural distributivity maps which are isomorphisms

$$\begin{aligned}
d^l &: (x \otimes y) \oplus (x' \otimes y) \rightarrow (x \oplus x') \otimes y, \\
d^r &: (x \otimes y) \oplus (x \otimes y') \rightarrow x \otimes (y \oplus y'),
\end{aligned}$$

that are defined for objects x and x' , y and y' of \mathcal{D}_c and $\mathcal{D}_{c'}$ respectively. The distributive maps are required to be maps in the category $\mathcal{D}_{c \otimes c'}$ and also to satisfy the coherences (a.1) – (a.5) where they make sense. We name these conditions (b.1) – (b.5). In addition, given $g : x \rightarrow y$ and $g' : x' \rightarrow y'$ morphisms over $f : c \rightarrow c'$, then we require the existence of a morphism $g \oplus g' : x \oplus x' \rightarrow y \oplus y'$ defined in a natural way; that is, if we have morphisms $h : y \rightarrow z$ and $h' : y' \rightarrow z'$ over $f' : c' \rightarrow c''$, then the following diagram is commutative

$$\begin{array}{ccc}
x \oplus x' & & \text{(b.6)} \\
\downarrow g \oplus g' & \searrow^{(h \circ h) \oplus (h' \circ g')} & \\
y \oplus y' & \xrightarrow{h \oplus h'} & z \oplus z'.
\end{array}$$

In the case that $f = \text{id} : c \rightarrow c$, then $g \oplus g' = g \oplus_c g'$. Also, we require this

addition to be associative; that is,

$$(g \oplus g') \oplus g'' = g \oplus (g' \oplus g'') \quad (\text{b.7})$$

for morphism g, g' and g'' over f . Moreover, we need the following diagram to commute

$$\begin{array}{ccc} x \oplus x' & \xrightarrow{g \oplus g'} & y \oplus y' \\ \gamma_c^\oplus \downarrow & & \downarrow \gamma_{c'}^\oplus \\ x' \oplus x & \xrightarrow{g' \oplus g} & y' \oplus y. \end{array} \quad (\text{b.8})$$

Finally, for each morphism $f : c \rightarrow c'$ in \mathcal{C} we can find a morphism $0_f : 0_c \rightarrow 0_{c'}$ over f , defined in a natural way in the sense that if $f' : c' \rightarrow c''$ is another morphism in \mathcal{C} then $0_{f'} \circ 0_f = 0_{f' \circ f}$ and the 0_f satisfy that

$$g \oplus 0_f = g = 0_f \oplus g. \quad (\text{b.9})$$

If $f = \text{id} : c \rightarrow c$, then $0_{\text{id}} = \text{id}_{0_c}$. Note that the naturality of the distributivity morphisms d^l and d^r means that given $g : x \rightarrow x_1$ and $g' : x' \rightarrow x'_1$ morphisms over $f : c \rightarrow c'$ and a morphism $h : y \rightarrow y'$ over $f' : d \rightarrow d'$, then the following diagram is commutative

$$\begin{array}{ccc} x \otimes y \oplus x' \otimes y & \xrightarrow{d^l} & (x \oplus x') \otimes y \\ g \otimes h \oplus g' \otimes h \downarrow & & \downarrow (g \oplus g') \otimes h \\ x_1 \otimes y_1 \oplus x'_1 \otimes y_1 & \xrightarrow{d^l} & (x_1 \oplus x'_1) \otimes y_1 \end{array} \quad (\text{b.10})$$

and similarly for d^r .

Remark 7 A fibered bipermutative category over the trivial category is just a bipermutative category as defined above.

For symmetric bimonoidal categories, it is well known that every symmetric bimonoidal category is equivalent to a bipermutative category. In our case this is also true; that is, every fibered symmetric bimonoidal category is equivalent to a fibered bipermutative category. We show this in the following theorem.

Theorem 8 Given a fibered symmetric bimonoidal category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$, we can find equivalent categories \mathcal{D}^s and \mathcal{C}^s together with a functor $\Lambda^s : \mathcal{D}^s \rightarrow \mathcal{C}^s$ that is a fibered bipermutative category. Moreover, there are equivalences $\Phi' : \mathcal{C} \rightarrow \mathcal{C}^s$ and $\Theta' : \mathcal{D} \rightarrow \mathcal{D}^s$ that are symmetric monoidal functors, and such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\Theta'} & \mathcal{D}^s \\ \Lambda \downarrow & & \downarrow \Lambda^s \\ \mathcal{C} & \xrightarrow{\Phi'} & \mathcal{C}^s. \end{array}$$

Proof: To begin, let us replace the symmetric monoidal category \mathcal{C} by an equivalent permutative category \mathcal{C}^s . This is a standard construction in category theory. The objects of \mathcal{C}^s are formal products of the form

$$\underline{c} = c_1 \boxtimes \cdots \boxtimes c_n,$$

where $n \geq 0$. When $n = 0$, $\underline{c} = ()$ is the empty product. Given such a sequence we define

$$\Phi(c_1 \boxtimes \cdots \boxtimes c_n) = c_1 \otimes (\cdots \otimes (c_{n-1} \otimes c_n) \cdots)$$

for $n > 0$, and for $n = 0$ we define

$$\Phi() = 1.$$

Here 1 is the unit of the symmetric monoidal category $(\mathcal{C}, \otimes, 1, \gamma^\otimes)$. The morphisms of \mathcal{C}^s are defined in the following way. Suppose

$$\begin{aligned} \underline{c} &= c_1 \boxtimes \cdots \boxtimes c_n, \\ \underline{d} &= d_1 \boxtimes \cdots \boxtimes d_m \end{aligned}$$

are two objects of \mathcal{C}^s . Then a morphism $f : \underline{c} \rightarrow \underline{d}$ in \mathcal{C}^s is a morphism in \mathcal{C} $f : \Phi(\underline{c}) \rightarrow \Phi(\underline{d})$. This way we obtain a category and $\Phi : \mathcal{C}^s \rightarrow \mathcal{C}$ is an equivalence of categories. The inverse $\Phi' : \mathcal{C} \rightarrow \mathcal{C}^s$ is defined on objects by $\Phi'(c) = c$, where on the right c is the string of objects of \mathcal{C} of length 1. Similarly Φ' is defined on morphisms.

Note that \mathcal{C}^s is a permutative category. Indeed, the product \boxtimes is given by juxtaposition; that is, given \underline{c} and \underline{d} objects of \mathcal{C}^s as before, then

$$\underline{c} \boxtimes \underline{d} = c_1 \boxtimes \cdots \boxtimes c_n \boxtimes d_1 \boxtimes \cdots \boxtimes d_m.$$

Similarly, if $f : \underline{c} \rightarrow \underline{c}'$ and $g : \underline{d} \rightarrow \underline{d}'$ are two morphisms in \mathcal{C}^s , then we define

$$f \boxtimes g : \underline{c} \boxtimes \underline{d} \rightarrow \underline{c}' \boxtimes \underline{d}'$$

to be the following composite in \mathcal{C}

$$\Phi(\underline{c} \boxtimes \underline{d}) \xrightarrow{\cong} \Phi(\underline{c}) \otimes \Phi(\underline{d}) \xrightarrow{f \otimes g} \Phi(\underline{c}') \otimes \Phi(\underline{d}') \xrightarrow{\cong} \Phi(\underline{c}' \boxtimes \underline{d}').$$

Here the outer isomorphisms, are the coherent isomorphisms in \mathcal{C} arising from a rearrangement of parenthesis in a given product.

Before continuing we introduce some notation. Given a formal sequence

$$\underline{x} = x_1 \boxtimes \cdots \boxtimes x_k$$

of objects of \mathcal{D} , then we denote

$$\Delta(\underline{x}) = \Delta(x_1 \boxtimes \cdots \boxtimes x_k) = x_1 \otimes (\cdots \otimes (x_{k-1} \otimes x_k) \cdots).$$

Thus $\Delta(x_1 \boxtimes \cdots \boxtimes x_k)$ is the object of \mathcal{D} obtained by multiplying the elements x_1, \dots, x_k in a consistent way.

On the other hand, since $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ is a fibered category, as explained in Theorem 13, for every morphism $f : c \rightarrow c'$ we can correspond a functor $f^* : \mathcal{D}_{c'} \rightarrow \mathcal{D}_c$ in such a way that the correspondence $c \mapsto \mathcal{D}_c, f \mapsto f^*$ is a contravariant lax 2-functor $\mathcal{C} \rightarrow \mathbf{Cat}$. We will fix from now on such an assignment.

Now we want to replace the category \mathcal{D} by an equivalent category \mathcal{D}^s . The objects of \mathcal{D}^s will be the set of formal sequences of the form

$$X = (c_1 \boxtimes \cdots \boxtimes c_n, (\underline{x}_1, f_1) \boxplus \cdots \boxplus (\underline{x}_m, f_m)),$$

where $n \geq 0, m \geq 0$, each \underline{x}_i is a formal product

$$\underline{x}_i = x_{i1} \boxtimes \cdots \boxtimes x_{ik_i}$$

and

$$f_i : \Phi(c_1 \boxtimes \cdots \boxtimes c_n) \rightarrow \Lambda(\Delta(\underline{x}_i))$$

is an isomorphism in \mathcal{C} .

For an object

$$X = (c_1 \boxtimes \cdots \boxtimes c_n, (\underline{x}_1, f_1) \boxplus \cdots \boxplus (\underline{x}_m, f_m)),$$

of \mathcal{D}^s we define $\Theta(X)$ in the following way. If $m > 0$,

$$\Theta(X) = f_1^*(\Delta(\underline{x}_1)) \oplus (\cdots \oplus (f_{m-1}^*(\Delta(\underline{x}_{m-1})) \oplus f_m^*(\Delta(\underline{x}_m)))).$$

When $m = 0$ and $n > 0$, then $X = (c_1 \boxtimes \cdots \boxtimes c_n,)$ and define

$$\Theta(X) = 0_{\Phi(c)}.$$

Finally, in \mathcal{D}^s we have an object of the form $X = ((), 1)$ which will be the multiplicative unit and we define

$$\Theta((), 1) = 1,$$

the unit of \mathcal{D} .

Note that $\Theta(X)$ is a well defined object in \mathcal{D} , this is because for every $1 \leq i \leq m$, each $f_i^*(\Delta(\underline{x}_i))$ is an object in $\mathcal{D}_{\Phi(c)}$ and addition is well defined on fibers. To define the morphisms of \mathcal{D}^s we use Θ as follows. Take two objects in \mathcal{D}^s

$$\begin{aligned} X &= (c_1 \boxtimes \cdots \boxtimes c_n, (\underline{x}_1, f_1) \boxplus \cdots \boxplus (\underline{x}_m, f_m)), \\ Y &= (d_1 \boxtimes \cdots \boxtimes d_r, (\underline{y}_1, g_1) \boxplus \cdots \boxplus (\underline{y}_s, g_s)). \end{aligned}$$

Then a morphism $f : X \rightarrow Y$ in \mathcal{D}^s , is precisely a morphism $f : \Theta(X) \rightarrow \Theta(Y)$ in \mathcal{D} . In this way we obtain a category \mathcal{D}^s and we trivially see that $\Theta : \mathcal{D}^s \rightarrow \mathcal{D}$

is a functor. It's easy to see that the $\Theta : \mathcal{D}^s \rightarrow \mathcal{D}$ defines an equivalence of categories with inverse $\Theta' : \mathcal{D} \rightarrow \mathcal{D}^s$ the functor that for an object x of \mathcal{D}_c corresponds $\Theta'(x) = (c, (x, \text{id}_c))$.

We continue now with the definition of the functor $\Lambda^s : \mathcal{D}^s \rightarrow \mathcal{C}^s$. Given an object of \mathcal{D}^s

$$X = (c_1 \boxtimes \cdots \boxtimes c_n, (\underline{x}_1, f_1) \boxplus \cdots \boxplus (\underline{x}_m, f_m)),$$

define

$$\Lambda^s(X) = c_1 \boxtimes \cdots \boxtimes c_n.$$

If $f : X \rightarrow Y$ is a morphism in \mathcal{D}^s , then $f : \Theta(X) \rightarrow \Theta(Y)$ is a morphism in \mathcal{D} and thus $\Lambda(f) : \Lambda(\Theta(X)) = \Phi(\Lambda^s(X)) \rightarrow \Lambda(\Theta(Y)) = \Phi(\Lambda^s(Y))$ is a morphism in \mathcal{C} . We define then

$$\Lambda^s(f) = \Lambda(f).$$

We want to see that $\Lambda^s : \mathcal{D}^s \rightarrow \mathcal{C}^s$ satisfies the required properties. To begin, note that by the definition we can easily see that

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\Theta'} & \mathcal{D}^s \\ \Lambda \downarrow & & \downarrow \Lambda^s \\ \mathcal{C} & \xrightarrow{\Phi'} & \mathcal{C}^s \end{array}$$

is a commutative diagram.

Our next step is to show that each fiber $\mathcal{D}_{\underline{c}}^s$ has the structure of a permutative category. Suppose then that

$$\begin{aligned} X &= (c_1 \boxtimes \cdots \boxtimes c_n, (\underline{x}_1, f_1) \boxplus \cdots \boxplus (\underline{x}_m, f_m)), \\ Y &= (c_1 \boxtimes \cdots \boxtimes c_n, (\underline{y}_1, g_1) \boxplus \cdots \boxplus (\underline{y}_s, g_s)) \end{aligned}$$

are two objects in the fiber $\mathcal{D}_{\underline{c}}^s$. Define

$$X \boxplus_{\underline{c}} Y = (c_1 \boxtimes \cdots \boxtimes c_n, (\underline{x}_1, f_1) \boxplus \cdots \boxplus (\underline{x}_m, f_m) \boxplus (\underline{y}_1, g_1) \boxplus \cdots \boxplus (\underline{y}_s, g_s)).$$

We want to see that this defines a permutative category on $\mathcal{D}_{\underline{c}}^s$. First of all, it is clear that $\boxplus_{\underline{c}}$ is strictly associative. In addition, if we denote by

$$0_{\underline{c}} = (c_1 \boxtimes \cdots \boxtimes c_n,$$

then it is clear that $0_{\underline{c}}$ is strict unit for $\boxplus_{\underline{c}}$. We need to find a symmetry isomorphism $\gamma_{\underline{c}}^{\boxplus}$. Given objects X and Y in the fiber $\mathcal{D}_{\underline{c}}^s$ as before, then we define a symmetry isomorphism

$$\gamma_{\underline{c}}^{\boxplus} : X \boxplus_{\underline{c}} Y \rightarrow Y \boxplus_{\underline{c}} X$$

to be the composite

$$\Theta(X \boxplus_{\underline{c}} Y) \xrightarrow{\cong} \Theta(X) \oplus \Theta(Y) \xrightarrow{\gamma_{c_1 \otimes (\dots (c_{n-1} \otimes c_n))}^{\oplus}} \Theta(Y) \oplus \Theta(X) \xrightarrow{\cong} \Theta(Y \boxplus_{\underline{c}} X)$$

where the outer isomorphisms are the coherent isomorphisms in $\mathcal{D}_{\Phi(\underline{c})}$ coming from the associativity of $\oplus_{\Phi(\underline{c})}$. Then using some coherent theory it follows that each $(\mathcal{D}_{\underline{c}}^s, \boxplus_{\underline{c}}, 0_{\underline{c}}, \gamma_{\underline{c}}^{\boxplus})$ is a permutative category.

Let us show now that the category \mathcal{D}^s has the structure of a permutative category under a product \boxtimes . We begin by defining the functor \boxtimes . Suppose then that

$$\begin{aligned} X &= (c_1 \boxtimes \dots \boxtimes c_n, (\underline{x}_1, f_1) \boxplus \dots \boxplus (\underline{x}_m, f_m)), \\ Y &= (d_1 \boxtimes \dots \boxtimes d_r, (\underline{y}_1, g_1) \boxplus \dots \boxplus (\underline{y}_s, g_s)) \end{aligned}$$

are two objects of \mathcal{D}^s with $n, m, r, s > 0$. We define

$$\begin{aligned} X \boxtimes Y &= \\ & (c_1 \boxtimes \dots \boxtimes c_n \boxtimes d_1 \boxtimes \dots \boxtimes d_r, (\underline{x}_1 \boxtimes \underline{y}_1, f_1 \boxtimes g_1) \boxplus \dots \boxplus (\underline{x}_1 \boxtimes \underline{y}_s, f_1 \boxtimes g_s) \boxplus \\ & \dots \boxplus (\underline{x}_m \boxtimes \underline{y}_1, f_m \boxtimes g_1) \boxplus \dots \boxplus (\underline{x}_m \boxtimes \underline{y}_s, f_m \boxtimes g_s)). \end{aligned}$$

Here, if

$$\begin{aligned} \underline{x}_i &= x_{i1} \boxtimes \dots \boxtimes x_{ik_i}, \\ \underline{y}_j &= y_{j1} \boxtimes \dots \boxtimes y_{jt_j} \end{aligned}$$

then

$$\underline{x}_i \boxtimes \underline{y}_j = x_{i1} \boxtimes \dots \boxtimes x_{ik_i} \boxtimes y_{j1} \boxtimes \dots \boxtimes y_{jt_j},$$

and $f_i \boxtimes g_j$ is defined to be the composite in \mathcal{D}

$$\begin{aligned} \Phi(\underline{c} \boxtimes \underline{d}) &\xrightarrow{\cong} \Phi(\underline{c}) \otimes \Phi(\underline{d}) \xrightarrow{f_i \otimes g_j} \Lambda(\Delta(\underline{x}_i)) \otimes \Lambda(\Delta(\underline{y}_j)) \xrightarrow{\lambda_{\Lambda}} \\ &\Lambda(\Delta(\underline{x}_i) \otimes \Delta(\underline{y}_j)) \xrightarrow{\cong} \Lambda(\Delta(\underline{x}_i \boxtimes \underline{y}_j)). \end{aligned}$$

The outer maps are the ones obtained by a rearrangement of parenthesis. Also we we define

$$((), 1) \boxtimes X = X = X \boxtimes ((), 1)$$

for all objects X of \mathcal{D}^s . Finally, if X is any object in $\mathcal{D}_{\underline{c}}^s$, then

$$\begin{aligned} 0_{\underline{d}} \boxtimes X &= 0_{\underline{d} \boxtimes \underline{c}} \\ X \boxtimes 0_{\underline{d}} &= 0_{\underline{c} \boxtimes \underline{d}}. \end{aligned}$$

In a similar way we define \boxtimes on morphisms of \mathcal{D}^s .

Let us show that this defines a permutative category on \mathcal{D}^s . We begin by

checking the associativity property. Suppose that

$$\begin{aligned} X &= (c_1 \boxtimes \cdots \boxtimes c_n, (\underline{x}_1, f_1) \boxplus \cdots \boxplus (\underline{x}_m, f_m)), \\ Y &= (d_1 \boxtimes \cdots \boxtimes d_r, (\underline{y}_1, g_1) \boxplus \cdots \boxplus (\underline{y}_s, g_s)), \\ Z &= (e_1 \boxtimes \cdots \boxtimes e_l, (\underline{z}_1, h_1) \boxplus \cdots \boxplus (\underline{z}_a, h_a)) \end{aligned}$$

are three objects of \mathcal{D}^s . Then by definition we have that

$$\begin{aligned} (X \boxtimes Y) \boxtimes Z &= (\underline{c} \boxtimes \underline{d} \boxtimes \underline{e}, (\underline{x}_1 \boxtimes \underline{y}_1 \boxtimes \underline{z}_1, (f_1 \boxtimes g_1) \boxtimes h_1) \boxplus \\ &\quad \cdots \boxplus (\underline{x}_m \boxtimes \underline{y}_s \boxtimes \underline{z}_a, (f_m \boxtimes g_s) \boxtimes h_a)), \end{aligned}$$

$$\begin{aligned} X \boxtimes (Y \boxtimes Z) &= (\underline{c} \boxtimes \underline{d} \boxtimes \underline{e}, (\underline{x}_1 \boxtimes \underline{y}_1 \boxtimes \underline{z}_1, f_1 \boxtimes (g_1 \boxtimes h_1)) \boxplus \\ &\quad \cdots \boxplus (\underline{x}_m \boxtimes \underline{y}_s \boxtimes \underline{z}_a, f_m \boxtimes (g_s \boxtimes h_a))) \end{aligned}$$

so the only question to be answered is whether or not $(f_u \boxtimes g_v) \boxtimes h_w$ and $f_u \boxtimes (g_v \boxtimes h_w)$ agree for all u, v and w . But these morphisms agree by coherence. Thus \boxtimes is strictly associative. By definition, $((\), 1)$ is a strict unit for \boxtimes . We construct now a symmetry isomorphism $\gamma^{\boxtimes} : X \boxtimes Y \rightarrow Y \boxtimes X$. To do so, take objects of \mathcal{D}^s of the form

$$\begin{aligned} X &= (c_1 \boxtimes \cdots \boxtimes c_n, (\underline{x}, f)), \\ Y &= (d_1 \boxtimes \cdots \boxtimes d_r, (\underline{y}, g)) \end{aligned}$$

where

$$\begin{aligned} \underline{x} &= x_1 \boxtimes \cdots \boxtimes x_m, \\ \underline{y} &= y_1 \boxtimes \cdots \boxtimes y_s. \end{aligned}$$

We will define first the symmetry isomorphism in this case

$$\gamma^{\boxtimes} : X \boxtimes Y \rightarrow Y \boxtimes X.$$

By definition, we have that

$$\begin{aligned} \Theta(X \boxtimes Y) &= (f \boxtimes g)^*(\Delta(\underline{x} \boxtimes \underline{y})), \\ \Theta(Y \boxtimes X) &= (g \boxtimes f)^*(\Delta(\underline{y} \boxtimes \underline{x})). \end{aligned}$$

We want to define a natural isomorphism

$$\gamma_{f,g}^{\boxtimes} : (f \boxtimes g)^*(\Delta(\underline{x} \boxtimes \underline{y})) \rightarrow (g \boxtimes f)^*(\Delta(\underline{y} \boxtimes \underline{x})).$$

By definition of a fibered category, there is a unique morphism $\gamma_{f,g}^{\boxtimes}$ making the following diagram commutative

$$\begin{array}{ccccc}
(f \boxtimes g)^*(\Delta(\underline{x} \boxtimes \underline{y})) & \xrightarrow{\quad} & \Delta(\underline{x} \boxtimes \underline{y}) & \xrightarrow{\Delta(\gamma^{\boxtimes})} & \Delta(\underline{y} \boxtimes \underline{x}) \\
\downarrow & \searrow \gamma_{f,g}^{\boxtimes} & \downarrow & \downarrow & \downarrow \\
& & (g \boxtimes f)^*(\Delta(\underline{y} \boxtimes \underline{x})) & \xrightarrow{\quad} & \Delta(\underline{y} \boxtimes \underline{x}) \\
\Phi(\underline{c} \boxtimes \underline{d}) & \xrightarrow{f \boxtimes g} & \Lambda(\Delta(\underline{x} \boxtimes \underline{y})) & \xrightarrow{\Lambda(\Delta(\gamma^{\boxtimes}))} & \Lambda(\Delta(\underline{y} \boxtimes \underline{x})) \\
\downarrow \gamma^{\boxtimes} & & \downarrow & & \downarrow \\
\Phi(\underline{d} \boxtimes \underline{c}) & \xrightarrow{g \boxtimes f} & \Lambda(\Delta(\underline{y} \boxtimes \underline{x})) & & \Lambda(\Delta(\underline{y} \boxtimes \underline{x}))
\end{array}$$

Because of the uniqueness condition, the morphism $\gamma_{f,g}^{\boxtimes}$ is natural and satisfies the required coherences. We extend the definition of the γ^{\boxtimes} to all the objects of \mathcal{D}^s in the following way. Suppose that

$$\begin{aligned}
X &= (c_1 \boxtimes \cdots \boxtimes c_n, (\underline{x}_1, f_1) \boxplus \cdots \boxplus (\underline{x}_m, f_m)), \\
Y &= (d_1 \boxtimes \cdots \boxtimes d_r, (\underline{y}_1, g_1) \boxplus \cdots \boxplus (\underline{y}_s, g_s))
\end{aligned}$$

are two general objects of \mathcal{D}^s . Then we define γ^{\boxtimes} to be the composite

$$\begin{aligned}
\Theta(X \boxtimes Y) &= \sum_{i=1}^m \sum_{j=1}^r (f_i \boxtimes g_j)^*(\Delta(\underline{x}_i \boxtimes \underline{y}_j)) \xrightarrow{\sum_{i=1}^m \sum_{j=1}^r \gamma_{f_i, g_j}^{\boxtimes}} \\
&= \sum_{i=1}^m \sum_{j=1}^r (g_j \boxtimes f_i)^*(\Delta(\underline{y}_j \boxtimes \underline{x}_i)) \xrightarrow{\cong} \sum_{j=1}^r \sum_{i=1}^m (g_j \boxtimes f_i)^*(\Delta(\underline{y}_j \boxtimes \underline{x}_i)) \\
&= \Theta(Y \boxtimes X)
\end{aligned}$$

Here by $\sum_{i=1}^m \sum_{j=1}^r x_{ij}$ we mean the sum with a consistent way of inserting parenthesis as before and the unlabeled isomorphism is the isomorphism obtained by rearranging the terms of summation using the isomorphism γ^{\oplus} . This way defined we see by using some coherence theory that $(\mathcal{D}^s, \boxtimes, ((, 1), \gamma^{\boxtimes}))$ has the structure of a permutative category.

Our next step is to construct distributivity maps

$$\begin{aligned}
d^l &: (X \boxtimes Y) \boxplus (X' \boxtimes Y) \rightarrow (X \boxplus X') \boxtimes Y, \\
d^r &: (X \boxtimes Y) \boxplus (X \boxtimes Y') \rightarrow X \boxtimes (Y \boxplus Y'),
\end{aligned}$$

wherever they make sense; that is, for objects X and X' of $\mathcal{D}_{\underline{c}}^s$ and Y and Y' of $\mathcal{D}_{\underline{d}}^s$. By a straight forward computation we can see that

$$d^l = \text{id} : (X \boxtimes Y) \boxplus (X' \boxtimes Y) = (X \boxplus X') \boxtimes Y.$$

On the other hand, if we write

$$\begin{aligned} X &= (c_1 \boxtimes \cdots \boxtimes c_n, (\underline{x}_1, f_1) \boxplus \cdots \boxplus (\underline{x}_m, f_m)), \\ Y &= (d_1 \boxtimes \cdots \boxtimes d_r, (\underline{y}_1, g_1) \boxplus \cdots \boxplus (\underline{y}_s, g_s)), \\ Y' &= (d_1 \boxtimes \cdots \boxtimes d_l, (\underline{y}'_1, g'_1) \boxplus \cdots \boxplus (\underline{y}'_{s'}, g'_{s'})) \end{aligned}$$

then we have that

$$\begin{aligned} X \boxtimes Y \boxplus X \boxtimes Y' &= \\ &= (\underline{c} \boxtimes \underline{d}, (\underline{x}_1 \boxtimes \underline{y}_1, f_1 \boxtimes g_1) \boxplus \cdots \boxplus (\underline{x}_m \boxtimes \underline{y}_s, f_m \boxtimes g_s) \\ &\quad \boxplus (\underline{x}_1 \boxtimes \underline{y}'_1, f_1 \boxtimes g'_1) \boxplus \cdots \boxplus (\underline{x}_m \boxtimes \underline{y}'_{s'}, f_m \boxtimes g'_{s'})) \end{aligned}$$

and

$$\begin{aligned} X \boxtimes (Y \boxplus Y') &= \\ &= (\underline{c} \boxtimes \underline{d}, (\underline{x}_1 \boxtimes \underline{y}_1, f_1 \boxtimes g_1) \boxplus \cdots \boxplus (\underline{x}_1 \boxtimes \underline{y}'_{s'}, f_1 \boxtimes g'_{s'}) \\ &\quad \boxplus (\underline{x}_m \boxtimes \underline{y}'_1, f_m \boxtimes g'_1) \boxplus \cdots \boxplus (\underline{x}_m \boxtimes \underline{y}'_{s'}, f_m \boxtimes g'_{s'})). \end{aligned}$$

Thus we define

$$d^r : (X \boxtimes Y) \boxplus (X \boxtimes Y') \rightarrow X \boxtimes (Y \boxplus Y')$$

as an iteration of $\gamma_{\underline{c} \boxtimes \underline{d}}^{\boxplus}$. By a trivial but long computation one can see that these distributivity maps are natural and satisfy the coherences of a fibered bipermutative category as in Definition 6.

To finish we need to show that we have addition on morphisms in \mathcal{D}^s over the same morphism in \mathcal{C}^s . Thus suppose that $\alpha : \underline{c} \rightarrow \underline{d}$ is a morphism in \mathcal{C}^s and $\beta : X \rightarrow Y$, $\delta : X' \rightarrow Y'$ are two morphism in \mathcal{D}^s over f , where X, X' are objects in $\mathcal{D}_{\underline{c}}^s$ and Y, Y' are objects in $\mathcal{D}_{\underline{d}}^s$. As noted before, we have a natural coherent isomorphism

$$\mu_{X, X'} : \Theta(X \boxplus_{\underline{c}} X') \xrightarrow{\cong} \Theta(X) \oplus \Theta(X').$$

This isomorphism is obtained by rearranging the parenthesis in the summations. With this in mind, we define $\alpha \boxplus \beta$ to be the following composite

$$\Theta(X \boxplus_{\underline{c}} X') \xrightarrow{\mu_{X, X'}} \Theta(X) \oplus \Theta(X') \xrightarrow{\alpha \oplus \beta} \Theta(Y) \oplus \Theta(Y') \xrightarrow{\mu_{Y, Y'}^{-1}} \Theta(Y \boxplus_{\underline{d}} Y').$$

This way defined we see that \boxplus is strictly associative and that

$$\begin{array}{ccc} X \boxplus_{\underline{c}} X' & \xrightarrow{\alpha \boxplus \beta} & Y \boxplus_{\underline{d}} Y' \\ \gamma_{\underline{c}}^{\boxplus} \downarrow & & \downarrow \gamma_{\underline{d}}^{\boxplus} \\ X' \boxplus_{\underline{c}} X & \xrightarrow{\beta \boxplus \alpha} & Y' \boxplus_{\underline{d}} Y \end{array}$$

is a commutative diagram.

In addition, note that by definition $\Theta(0_c) = 0_{\Phi(c)}$ and we have a coherent morphism $0_f : 0_{\Phi(c)} \rightarrow 0_{\Phi(d)}$ in \mathcal{D} , therefore we define a morphism $0_f : 0_c \rightarrow 0_d$ in \mathcal{D}^s to be the morphism $0_f : 0_{\Phi(c)} \rightarrow 0_{\Phi(d)}$ in \mathcal{D} . The coherences satisfied by 0_f in \mathcal{D} imply that

$$0_f \boxplus \alpha = \alpha = \alpha \boxplus 0_f.$$

Finally, since $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ is a fibered category, it follows easily that $\Lambda^s : \mathcal{D}^s \rightarrow \mathcal{C}^s$ is also a fibered category. We conclude then that $\Lambda^s : \mathcal{D}^s \rightarrow \mathcal{C}^s$ is a fibered bipermutative category. This proves the theorem. \square

3 Fibered categories

In this section we review briefly some aspects of fibered categories. In particular, we recall the fact that there is a one to one correspondence between fibered categories and contravariant functors to the category of small categories. We use this construction as motivation for our study of fibered bipermutative categories. Indeed, as mentioned before, using this correspondence, given a fibered bipermutative category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ we can assign a functor $\mathcal{C}^{op} \rightarrow \mathbb{P}$. However, this functor does not capture all the information present in a fibered bipermutative category and because of this we need to enlarge the category \mathcal{C}^{op} . Thus we show that for a given fibered bipermutative category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ we can correspond a functor $\Psi : \mathcal{A} \rightarrow \mathbb{P}$, where \mathcal{A} is a wreath product for a functor $(\mathcal{C}^{op})^* : \mathbf{Inj} \rightarrow \mathbb{P}$. The category \mathcal{C}^{op} canonically includes into \mathcal{A} and under this inclusion, the functor Ψ recovers the construction for fibered categories. Moreover, the functor Ψ recovers the multiplicative structure on \mathcal{D} . The goal of this section is to construct the functor Ψ . We begin by recalling the definition of a fibered category.

Definition 9 *If $F : \mathcal{D} \rightarrow \mathcal{C}$ is a functor and c an object of \mathcal{C} , we denote by \mathcal{D}_c the fiber $F^{-1}(c)$; that is, the subcategory of \mathcal{D} whose objects are the objects d such that $F(d) = c$ and the morphisms are the morphisms mapping to the identity of c .*

Definition 10 *A fibered category over \mathcal{C} is a functor $F : \mathcal{D} \rightarrow \mathcal{C}$ that satisfies the following properties:*

- *for any morphism $f : c \rightarrow c'$ in \mathcal{C} and any object d' of $\mathcal{D}_{c'}$, we can find an object d of \mathcal{D}_c and a morphism $g : d \rightarrow d'$ such that $F(g) = f$,*
- *given any pair of morphisms $g : d \rightarrow d'$ and $g' : d'' \rightarrow d'$ in \mathcal{D} , let $f = F(g) : c \rightarrow c'$ and $f' = F(g') : c'' \rightarrow c'$ their image in \mathcal{C} . Then for any $\tilde{f} : c'' \rightarrow c$ such that $f\tilde{f} = f'$, there is a unique morphism $\tilde{g} : d \rightarrow d''$ such that $g\tilde{g} = g'$ and $F(\tilde{g}) = \tilde{f}$.*

In general for a functor $F : \mathcal{D} \rightarrow \mathcal{C}$ there is the possibility that there is an isomorphism $f : c \rightarrow c'$ with \mathcal{D}_c and $\mathcal{D}_{c'}$ not equivalent categories. This does not happen in the case of fibered categories as any such isomorphism f induces an equivalence of categories $f^* : \mathcal{D}_{c'} \rightarrow \mathcal{D}_c$.

Definition 11 Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a functor and $g : d \rightarrow d'$ an arrow in \mathcal{D} . Take $g' : d'' \rightarrow d'$ in \mathcal{D} , and let $f = F(g) : c \rightarrow c'$, $f' = F(g') : c'' \rightarrow c'$ their image in \mathcal{C} . We say that g is a cartesian arrow if for any $\tilde{f} : c'' \rightarrow c$ such that $f\tilde{f} = f'$, there is a unique morphism $\tilde{g} : d'' \rightarrow d$ such that $g\tilde{g} = g'$ and $F(\tilde{g}) = \tilde{f}$. Such a cartesian arrow g over f is called a pullback of f .

This definition is summarized by the following diagram

$$\begin{array}{ccccc}
 d'' & & \xrightarrow{g'} & & d' \\
 \downarrow & \nearrow \tilde{g} & & \searrow g & \downarrow \\
 c'' & & & & c' \\
 \downarrow & \searrow \tilde{f} & & \searrow f & \downarrow \\
 c & & \xrightarrow{f} & & c'
 \end{array}$$

Thus a fibered category over \mathcal{C} is precisely a functor $F : \mathcal{D} \rightarrow \mathcal{C}$ such that given any morphism $f : c \rightarrow c'$ in \mathcal{C} and d' any object in $\mathcal{D}_{c'}$, there exists an object d in \mathcal{D}_c and a cartesian arrow $g : d \rightarrow d'$ such that $F(g) = f$.

Example: If u is an object of \mathcal{C} , then the comma category \mathcal{C}/u is fibered category over \mathcal{C} by considering $\theta_u : \mathcal{C}/u \rightarrow \mathcal{C}$ defined on objects by $\theta_u(f : v \rightarrow u) = v$ and if $\phi : (g : w \rightarrow u) \rightarrow (f : v \rightarrow u)$ is a morphism in \mathcal{C}/u ; that is, $\phi : w \rightarrow v$ is such that $f\phi = g$, then $\theta_u(\phi) = \phi$.

Definition 12 If $F : \mathcal{D} \rightarrow \mathcal{C}$ and $G : \mathcal{E} \rightarrow \mathcal{C}$ are two fibered categories over \mathcal{C} , then a morphism of fibered categories, $\phi : \mathcal{D} \rightarrow \mathcal{E}$, is a functor such that $F = G\phi$ and ϕ sends cartesian arrows to cartesian arrows.

Given two fibered categories $F : \mathcal{D} \rightarrow \mathcal{C}$ and $G : \mathcal{E} \rightarrow \mathcal{C}$, we denote by $\text{Hom}_{\mathcal{C}}(\mathcal{D}, \mathcal{E})$ the category whose objects are the morphisms of fibered categories $\phi : \mathcal{D} \rightarrow \mathcal{E}$. If $\phi, \psi : \mathcal{D} \rightarrow \mathcal{E}$ are objects in $\text{Hom}_{\mathcal{C}}(\mathcal{D}, \mathcal{E})$, then a morphism $\alpha : \phi \rightarrow \psi$ is a base point preserving natural transformation; that is, a natural transformation α such that for every object d of \mathcal{D} over c , the morphism α_d is a morphism in the category \mathcal{E}_c .

Note that the definition of a fibered category over \mathcal{C} states that given any morphism $f : c \rightarrow c'$ and any object d' of $\mathcal{D}_{c'}$, then we can choose an object f^*d' over c and a morphism $\eta : f^*d' \rightarrow d'$ over f . We call such a morphism a pullback of f . The morphism η is uniquely determined up to composition with an isomorphism in the fiber \mathcal{D}_c . So for each f and each object d' over c' we can fix such a pullback η over f . Also, if $\alpha : d' \rightarrow e'$ is a morphism in $\mathcal{D}_{c'}$, then by the uniqueness part in the definition of a cartesian arrow, we see that

there is a unique morphism $f^*\alpha$ in the category \mathcal{D}_c such that the following diagram commutes

$$\begin{array}{ccc} f^*d' & \xrightarrow{\eta_{d'}} & d' \\ f^*\alpha \downarrow & & \downarrow \alpha \\ f^*e' & \xrightarrow{\eta_{e'}} & e' \end{array}$$

This defines a functor $f^* : \mathcal{D}_{c'} \rightarrow \mathcal{D}_c$. Notice that if f and g are composable morphisms in \mathcal{C} , then $(fg)^*$ does not necessarily agree with g^*f^* . However, we can find a canonical isomorphism between $(fg)^*$ and g^*f^* . Thus for a fibered category $F : \mathcal{D} \rightarrow \mathcal{C}$, and a choice of pullbacks for each morphism f in \mathcal{C} , we can associate the following correspondence:

- for an object c of \mathcal{C} we associate the category \mathcal{D}_c ,
- for a morphism $f : c \rightarrow c'$ we associate the functor $f^* : \mathcal{D}_{c'} \rightarrow \mathcal{D}_c$.

At a first glance it would seem that this defines a functor $\mathcal{C}^{op} \rightarrow \mathbf{Cat}$, but as we just pointed out, this is not the case. However, \mathbf{Cat} has the structure of a 2-category and the previous assignment gives rise to a lax 2-functor and we have the following theorem.

Theorem 13 *The above assignment defines a one to one correspondence between isomorphism classes of fibered categories with a choice of pullbacks and isomorphisms of contravariant lax 2-functors $\mathcal{C} \rightarrow \mathbf{Cat}$.*

Proof: See (16, 3.1.2 and 3.1.3). \square

In fact up to isomorphism, we can replace any given fibered category $F : \mathcal{D} \rightarrow \mathcal{C}$ with an isomorphic fibered category $F' : \mathcal{D}' \rightarrow \mathcal{C}$ such that the corresponding assignment $\mathcal{C}^{op} \rightarrow \mathbf{Cat}$ is indeed a functor. Conversely, given any functor $\mathcal{C}^{op} \rightarrow \mathbf{Cat}$, we can associate a fibered category $F : \mathcal{D} \rightarrow \mathcal{C}$. This defines a one to one correspondence between isomorphism classes of fibered categories and isomorphism classes of functors $\mathcal{C}^{op} \rightarrow \mathbf{Cat}$. We show this in the following theorem.

Theorem 14 *There is a one to one correspondence between isomorphism classes of fibered categories and isomorphism classes of functors $\mathcal{C}^{op} \rightarrow \mathbf{Cat}$.*

Proof: We only sketch the proof of this theorem. For the details we refer the reader to (16, Chapter 3). Suppose first that $F : \mathcal{D} \rightarrow \mathcal{C}$ is a fibered category. We want to define a functor $f : \mathcal{C}^{op} \rightarrow \mathbf{Cat}$. For an object u define $f(u)$ to be the category $\text{Hom}_{\mathcal{C}}(\mathcal{C}/u, \mathcal{D})$. Given a morphism $g : u \rightarrow v$ in \mathcal{C} , by composing with g we obtain a morphism of fibered categories $g_* : \mathcal{C}/u \rightarrow \mathcal{C}/v$. This induces $f(g) : \text{Hom}_{\mathcal{C}}(\mathcal{C}/v, \mathcal{D}) \rightarrow \text{Hom}_{\mathcal{C}}(\mathcal{C}/u, \mathcal{D})$. It follows easily that this defines a functor $f : \mathcal{C}^{op} \rightarrow \mathbf{Cat}$. On the other hand, given a functor $f : \mathcal{C}^{op} \rightarrow \mathbf{Cat}$ we can construct a fibered category $F : \mathcal{D} \rightarrow \mathcal{C}$ in the following

way. The objects of \mathcal{D} are the pairs (x, c) , where c is an object in \mathcal{C} and x is an object in $F(c)$. If (x, c) and (y, d) are two such pairs, then a morphism in \mathcal{D} from (x, c) to (y, d) is a pair (α, f) , where $f : c \rightarrow d$ is a morphism in \mathcal{C} and $\alpha : x \rightarrow F(f)y$ is a morphism in the category $F(c)$. If $(\alpha, f) : (x, c) \rightarrow (y, d)$, $(\beta, g) : (y, d) \rightarrow (z, e)$, then

$$(\beta, g) \circ (\alpha, f) := (F(f)(\beta) \circ \alpha, g \circ f) : (x, c) \rightarrow (z, e).$$

It's easy to see that this defines a category. In addition we have a functor $F : \mathcal{D} \rightarrow \mathcal{C}$ that sends a pair (x, c) to c and a morphism (α, f) to f . This makes \mathcal{D} into a fibered category over \mathcal{C} whose fiber \mathcal{D}_c over an object c of \mathcal{C} is naturally isomorphic to the category $F(c)$. We leave to the reader to show that this is indeed a one to one correspondence. \square

We can apply this procedure to a bipermutative category on fibers $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$. This way we obtain a functor $f : \mathcal{C}^{op} \rightarrow \mathbf{Cat}$ that has the further property that each $f(u)$ has the structure of a permutative category and the functor f can be seen as a functor $f : \mathcal{C}^{op} \rightarrow \mathbb{P}$. However, this functor does not behaves well under the multiplicative structure on \mathcal{D} . We show however, that the fibered category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ determines and is determined by a functor $\Psi : \mathcal{A} \rightarrow \mathbb{P}$ that behaves well under multiplication. Here \mathcal{A} is a category that is obtained as a wreath product $\mathbf{Inj} f(\mathcal{C}^{op})^*$ for a functor $(\mathcal{C}^{op})^* : \mathbf{Inj} \rightarrow \mathbb{P}$. The category \mathcal{C} naturally embeds into \mathcal{A} and under this embedding we recover the functor $f : \mathcal{C}^{op} \rightarrow \mathbb{P}$ arising from the fibered category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$. After the construction of the functor Ψ , we show in theorem 26 that it satisfies conditions (c.1) – (c.14) of theorem 25. We will explain how to construct the functor $\Psi : \mathcal{A} \rightarrow \mathbb{P}$ in what follows.

We begin by constructing the category \mathcal{A} . This is a straight forward generalization of (4, Definition 5.1). Let us denote by \mathbf{Inj} the category whose object set is the set of integers $n \geq 0$, where we identify the integer n with the set $\underline{n} = \{1, \dots, n\}$. The morphism set from n to m is the set of injective functions from $\underline{n} = \{1, \dots, n\}$ to $\underline{m} = \{1, \dots, m\}$ with composition the composition of functions. Define the functor

$$(\mathcal{C}^{op})^* : \mathbf{Inj} \rightarrow \mathbb{P}$$

such that $(\mathcal{C}^{op})^*(n) = (\mathcal{C}^{op})^n$. If $q : \underline{n} \rightarrow \underline{m}$ is a morphism in \mathbf{Inj} ; that is, $q : \underline{n} \rightarrow \underline{m}$ is an injective map, then $(\mathcal{C}^{op})^*(q) : (\mathcal{C}^{op})^n \rightarrow (\mathcal{C}^{op})^m$ is the functor that for an n -tuple $\underline{u} = (u_1, \dots, u_n)$ corresponds $q_*\underline{u} = (u'_1, \dots, u'_m)$, where

$$u'_j = \begin{cases} u_i & \text{if } q^{-1}(j) = \{i\}, \\ 1 & \text{if } q^{-1}(j) = \emptyset, \end{cases}$$

and if $\underline{f} = (f_1, \dots, f_n) : \underline{u} \rightarrow \underline{v}$ is a morphism in $(\mathcal{C}^{op})^n$, then $\mathcal{C}^*(\underline{f}) = \underline{f}' : q_*\underline{u} \rightarrow q_*\underline{v}$, where $\underline{f}' = (f'_1, \dots, f'_m)$ and

$$f'_j = \begin{cases} f_i & \text{if } q^{-1}(j) = \{i\}, \\ \text{id}_1 & \text{if } q^{-1}(j) = \emptyset. \end{cases}$$

Then associated to this functor, there is a wreath product category $\mathcal{A} = \mathbf{Inj} f(\mathcal{C}^{op})^*$. More explicitly, the objects of \mathcal{A} are the sequences of the form (u_1, \dots, u_n) where $n \geq 0$ and a morphism from $\underline{u} = (u_1, \dots, u_n)$ to $\underline{v} = (v_1, \dots, v_m)$ is a pair (q, \underline{f}) , where $q : \underline{n} \rightarrow \underline{m}$ is an injection and $\underline{f} : q_*\underline{u} \rightarrow \underline{v}$ is a morphism in $(\mathcal{C}^{op})^n$. Note that \mathcal{A} is a permutative category by concatenation. Indeed, for objects $\underline{u} = (u_1, \dots, u_n)$ and $\underline{v} = (v_1, \dots, v_m)$, we can define $\underline{u} \odot \underline{v} = (u_1, \dots, u_n, v_1, \dots, v_m)$ and similarly on morphisms. The unit of \mathcal{A} is the empty tuple $()$.

Suppose that $\underline{u} = (u_1, \dots, u_n)$ is an object of \mathcal{A} . We will denote by $\mathcal{C}/\underline{u}$ the product category $\mathcal{C}/u_1 \times \dots \times \mathcal{C}/u_n$. We can see $\mathcal{C}/\underline{u}$ as a category over \mathcal{C} by defining the functor $\theta_{\underline{u}} : \mathcal{C}/\underline{u} \rightarrow \mathcal{C}$ as follows. Given objects $f_i : w_i \rightarrow u_i$ in \mathcal{C}/u_i , then $\theta(f_1, \dots, f_n) = w_1 \otimes \dots \otimes w_n$. Also, if $\phi_i : w_i \rightarrow w'_i$ is a morphism in \mathcal{C}/u_i , then $\theta(\phi_1, \dots, \phi_n) = \phi_1 \otimes \dots \otimes \phi_n$. As explained above, this makes $\theta_{u_1} : \mathcal{C}/u_1 \rightarrow \mathcal{C}$ into a fibered category in the case where $n = 1$. This is not the case in general for $n > 1$. When $\underline{u} = ()$ is the empty tuple, then we will understand by $\mathcal{C}/()$ the trivial category with only object $\text{id} : 1 \rightarrow 1$ and over \mathcal{C} . With this convention then the category $\text{Hom}_{\mathcal{C}}(\mathcal{C}/(), \mathcal{D})$ is isomorphic to the category \mathcal{D}_1 . Note that since we are assuming that every morphism in \mathcal{C} is an isomorphism, then a morphism of fibered categories $\mathcal{C}/u \rightarrow \mathcal{D}$ is just a functor $F : \mathcal{C}/u \rightarrow \mathcal{D}$ such that $\Lambda \circ F = \theta_u$.

Let us construct now the functor $\Psi : \mathcal{A} \rightarrow \mathbb{P}$. Take $\underline{u} = (u_1, \dots, u_n)$ an object of \mathcal{A} . Define $\Psi(\underline{u})$ to be the category whose objects are the functors $F : \mathcal{C}/\underline{u} \rightarrow \mathcal{D}$ such that $\Phi \circ F = \theta_{\underline{u}}$ and such that F is of the form

$$F = \sum_{i=1}^r F_{i1} \otimes \dots \otimes F_{in},$$

where each F_{ij} is an object of $\text{Hom}_{\mathcal{C}}(\mathcal{C}/u_i, \mathcal{D})$. This means that given an object $\underline{f} = (f_1, \dots, f_n)$ in $\mathcal{C}/\underline{u}$, then

$$F(\underline{f}) = F_{11}(f_1) \otimes \dots \otimes F_{1n}(f_n) \oplus \dots \oplus F_{r1}(f_1) \otimes \dots \otimes F_{rn}(f_n)$$

and given $\underline{\phi} = (\phi_1, \dots, \phi_n)$ a morphism in $\mathcal{C}/\underline{u}$, then

$$F(\underline{\phi}) = F_{11}(\phi_1) \otimes \dots \otimes F_{1n}(\phi_n) \oplus \dots \oplus F_{r1}(\phi_1) \otimes \dots \otimes F_{rn}(\phi_n).$$

This defines the objects of $\Psi(\underline{u})$. Given

$$F = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in} \text{ and } G = \sum_{j=1}^s G_{j1} \otimes \cdots \otimes G_{jn}$$

two such functors, then a morphism $\alpha : F \rightarrow G$ in $\Psi(\underline{u})$ is a base point preserving natural transformation $F \rightarrow G$; that is, α is a natural transformation such that for an object \underline{f} of $\mathcal{C}/\underline{u}$, with $f_i : w_i \rightarrow u_i$, then $\alpha_{\underline{f}} : F(\underline{f}) \rightarrow G(\underline{f})$ is a morphism in $\mathcal{D}_{\otimes w_i}$. As the composition of a base preserving natural transformation is also a base preserving natural transformation we obtain this way a well defined category $\Psi(\underline{u})$. When $\underline{u} = ()$ is the empty tuple, then $\Psi(())$ is then the category whose objects are functors $F : \mathcal{C}/() \rightarrow \mathcal{D}$ such that $F = \sum_i^n F_i$ with F_i an object of $\text{Hom}_{\mathcal{C}}(\mathcal{C}/(), \mathcal{D})$ and with morphisms the base preserving natural transformations. This category is canonically isomorphic to \mathcal{D}_1 under the isomorphism $F \mapsto F(\text{id} : 1 \rightarrow 1)$.

We want to show that if $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ is a fibered bipermutative category, then the category $\Psi(\underline{u})$ has the structure of a permutative category. To do so, we need to define a functor $\oplus : \Psi(\underline{u}) \times \Psi(\underline{u}) \rightarrow \Psi(\underline{u})$ and show that it satisfies the respective coherence. Suppose then that $F = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in}$ and $G = \sum_{j=1}^s G_{j1} \otimes \cdots \otimes G_{jn}$ are two objects of $\Psi(\underline{u})$. Define

$$F \oplus G = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in} \oplus \sum_{j=1}^s G_{j1} \otimes \cdots \otimes G_{jn}.$$

This way defined we see that given \underline{f} an object in $\mathcal{C}/\underline{u}$

$$(F \oplus G)(\underline{f}) = F(\underline{f}) \oplus G(\underline{f}).$$

Given $\underline{\phi} = (\phi_1, \dots, \phi_n) : \underline{f} \rightarrow \underline{f}'$ a morphism in $\mathcal{C}/\underline{u}$, then

$$(F \oplus G)(\underline{\phi}) = F(\underline{\phi}) \oplus G(\underline{\phi}) : F(\underline{f}) \oplus G(\underline{f}) \rightarrow F(\underline{f}') \oplus G(\underline{f}')$$

is well defined as both $F(\underline{\phi})$ and $G(\underline{\phi})$ are morphisms over $\otimes \phi_i$ and we can add two such morphisms. It is easy to see that since the sum of morphisms over $\otimes \phi_i$ is natural then $F \oplus G$ is also an object of $\Psi(\underline{u})$. This defines \oplus on the objects of $\Psi(\underline{u})$. Suppose now that $\alpha : F \rightarrow G$ and $\beta : H \rightarrow K$ are two morphisms in $\Psi(\underline{u})$. Given \underline{f} an object in $\mathcal{C}/\underline{u}$, define

$$(\alpha \oplus \beta)_{\underline{f}} = \alpha_{\underline{f}} \oplus \beta_{\underline{f}} : F(\underline{f}) \oplus H(\underline{f}) \rightarrow G(\underline{f}) \oplus K(\underline{f}).$$

It follows at once that this way defined, $\alpha \oplus \beta$ is a base preserving natural transformation and that $\oplus : \Psi(\underline{u}) \times \Psi(\underline{u}) \rightarrow \Psi(\underline{u})$ is a functor. We claim that \oplus is strictly associative. Indeed, if F, G and H are objects of $\Psi(\underline{u})$, then both $(F \oplus G) \oplus H$ and $F \oplus (G \oplus H)$ agree as functors as by assumption each \oplus_u is strictly associative and the addition of morphisms is also strictly associative.

Next we define an additive unit. We need to define an object $\mathbb{O}_{\underline{u}}$ of $\mathcal{C}/\underline{u}$. To do so, given u an object of \mathcal{C} we define \mathbb{O}_u to be the object of $\text{Hom}_{\mathcal{C}}(\mathcal{C}/u, \mathcal{D})$ defined by as follows. If $f : w \rightarrow u$ is an object in \mathcal{C}/u , then $\mathbb{O}(f) = 0_w$, where 0_w is the unit of the permutative category \mathcal{D}_w . If $\phi : w \rightarrow w'$ is a morphism in \mathcal{C}/u then $\mathbb{O}(\phi) = 0_\phi : 0_w \rightarrow 0_{w'}$ is the coherent morphism over ϕ as whose existence is guaranteed by hypothesis. Then we define $\mathbb{O}_{\underline{u}} = \mathbb{O}_{u_1} \otimes \cdots \otimes \mathbb{O}_{u_n}$. This way defined $\mathbb{O}_{\underline{u}}$ is an object of $\Psi(\underline{u})$. We claim this is a strict unit; that is, we want to see that for every F object of $\Psi(\underline{u})$

$$F \oplus \mathbb{O}_{\underline{u}} = F = \mathbb{O}_{\underline{u}} \oplus F.$$

Indeed, if \underline{f} is an object of $\mathcal{C}/\underline{u}$, then by definition

$$\begin{aligned} (F \oplus \mathbb{O}_{\underline{u}})(\underline{f}) &= F(\underline{f}) \oplus \mathbb{O}_{\underline{u}}(\underline{f}) = F(\underline{f}) \oplus 0_{\otimes w_i} = F(\underline{f}) \\ (\mathbb{O}_{\underline{u}} \oplus F)(\underline{f}) &= \mathbb{O}_{\underline{u}}(\underline{f}) \oplus F(\underline{f}) = 0_{\otimes w_i} \oplus F(\underline{f}) = F(\underline{f}) \end{aligned}$$

and similarly on morphisms.

We also need to construct a symmetry isomorphism. Suppose then that F and G are two objects of $\Psi(\underline{u})$. Define $\gamma : F \oplus G \rightarrow G \oplus F$ to be the transformation that for each object \underline{f} of $\mathcal{C}/\underline{u}$ assigns

$$\gamma_{\underline{f}} = \gamma_{\otimes w_i}^{\oplus} : F(\underline{f}) \oplus G(\underline{f}) \rightarrow G(\underline{f}) \oplus F(\underline{f}).$$

This is a natural transformation as addition commutes with the different $\gamma_{\underline{u}}^{\oplus}$ and is base preserving by definition. In order to conclude that $(\Psi(\underline{u}), \oplus, \gamma)$ is a permutative category we need to verify that the following diagrams are commutative

$$\begin{array}{ccc} F \oplus G & \xrightarrow{=} & F \oplus G \\ & \searrow \gamma & \nearrow \gamma \\ & G \oplus F, & \end{array}$$

$$\begin{array}{ccc} F \oplus G \oplus H & \xrightarrow{\gamma} & H \oplus F \oplus G \\ & \searrow id \oplus \gamma & \nearrow \gamma \oplus id \\ & F \oplus H \oplus G. & \end{array}$$

But the commutativity of these diagrams follows as we have the commutativity of the corresponding diagrams when we evaluate on objects and morphisms, therefore $\Psi(\underline{u})$ is an object of \mathbb{P} .

We now want to show that we can extend the definition of Ψ as to get a functor $\Psi : \mathcal{A} \rightarrow \mathbb{P}$. To start, for a given morphism $f : u \rightarrow v$ in \mathcal{C} we define a functor

$$\begin{aligned} f^* : \text{Hom}_{\mathcal{C}}(\mathcal{C}/v, \mathcal{D}) &\rightarrow \text{Hom}_{\mathcal{C}}(\mathcal{C}/u, \mathcal{D}) \\ F &\mapsto f^* F \\ \alpha &\mapsto f^* \alpha. \end{aligned}$$

The functor f^*F is defined as follows. For an object $g : w \rightarrow u$ of \mathcal{C}/u $f^*F(g) = F(f \circ g)$ and if $\phi : f \rightarrow f'$ is a morphism in \mathcal{C}/u then ϕ can be seen as a morphism in \mathcal{C}/v between $f \circ g$ and $f' \circ g$, thus we define $f^*F(\phi) = F(\phi)$. Similarly we define $f^*\alpha$ for a base preserving natural transformation $\alpha : F \rightarrow G$. Suppose now that $\underline{u} = (u_1, \dots, u_n)$ and $\underline{v} = (v_1, \dots, v_m)$ are two objects of \mathcal{A} and that $(q, \underline{f}) : \underline{u} \rightarrow \underline{v}$ is a morphism in \mathcal{A} , thus $\underline{f} : q_*\underline{u} = \underline{u}' \rightarrow \underline{v}$ is a morphism in $(\mathcal{C}^{op})^m$; that is, $f_i : v_i \rightarrow u'_i$ is a morphism in \mathcal{C} . We want to define a lax $_*$ map $\Psi(q, \underline{f}) : \Psi(\underline{u}) \rightarrow \Psi(\underline{v})$. Suppose first that $m = n$ and thus $q = \sigma : \underline{n} \rightarrow \underline{n}$ is a permutation. Then each $f_i : v_i \rightarrow u_{\sigma^{-1}(i)}$ is a morphism in \mathcal{C} . Take

$$F = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in},$$

an object of $\mathcal{C}/\underline{u}$, where each F_{ij} is an object of $\text{Hom}_{\mathcal{C}}(\mathcal{C}/u_i, \mathcal{D})$. Define

$$\Psi(\sigma, \underline{f})F = \sum_{i=1}^r f_1^* F_{i\sigma^{-1}(1)} \otimes \cdots \otimes f_n^* F_{i\sigma^{-1}(n)}.$$

By definition it is clear that this is an object in $\Psi(\underline{v})$. Suppose now that $\alpha : F \rightarrow G$ is a morphism in $\Psi(\underline{u})$, where

$$F = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in} \quad \text{and} \quad G = \sum_{j=1}^s G_{j1} \otimes \cdots \otimes G_{jn}.$$

Then we need to define

$$\Psi(\sigma, \underline{f})\alpha : \Psi(\sigma, \underline{f})F \rightarrow \Psi(\sigma, \underline{f})G.$$

For each $1 \leq i \leq n$, the symmetry isomorphism γ^{\otimes} determines a coherent natural isomorphism

$$\tau_{\sigma, f_1^* F_{i\sigma^{-1}(1)}, \dots, f_n^* F_{i\sigma^{-1}(n)}} : f_1^* F_{i\sigma^{-1}(1)} \otimes \cdots \otimes f_n^* F_{i\sigma^{-1}(n)} \rightarrow f_{\sigma(1)}^* F_{i1} \otimes \cdots \otimes f_{\sigma(n)}^* F_{in}$$

and similarly for G . Since Φ is assumed to be a strict map, then for each object $\underline{g} = (g_1, \dots, g_n)$, with $g_i : w_i \rightarrow v_i$ of $\mathcal{C}/\underline{v}$ the morphism

$$\begin{aligned} & \tau_{\sigma, f_1^* F_{i\sigma^{-1}(1)}, \dots, f_n^* F_{i\sigma^{-1}(n)}}(\underline{g}) \\ & : f_1^* F_{i\sigma^{-1}(1)}(g_1) \otimes \cdots \otimes f_n^* F_{i\sigma^{-1}(n)}(g_n) \rightarrow f_1^* F_{i\sigma(1)}(g_{\sigma(1)}) \otimes \cdots \otimes f_n^* F_{i\sigma(n)}(g_{\sigma(n)}) \end{aligned}$$

is a morphism over $\tau_{\sigma, w_1, \dots, w_n} : w_1 \otimes \cdots \otimes w_n \rightarrow w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}$ which is the coherent isomorphism in \mathcal{C} provided by γ^{\otimes} .

Then we define $(\Psi(\sigma, \underline{f})\alpha)_{\underline{g}}$, by the following commutative diagram

$$\begin{array}{ccc}
\sum_{i=1}^r f_1^* F_{i\sigma^{-1}(1)}(g_1) \otimes \cdots \otimes f_n^* F_{i\sigma^{-1}(n)}(g_n) & & \\
\downarrow a & \searrow (\Psi(\sigma, \underline{f})\alpha)_{\underline{g}} & \\
\sum_{j=1}^s f_1^* G_{j\sigma^{-1}(1)}(g_1) \otimes \cdots \otimes f_n^* G_{j\sigma^{-1}(n)}(g_{\sigma(n)}) & & \\
\downarrow & & \uparrow b \\
\sum_{i=1}^r f_{\sigma(1)}^* F_{i1}(g_{\sigma(1)}) \otimes \cdots \otimes f_{\sigma(n)}^* F_{in}(g_{\sigma(n)}) & & \\
\searrow \alpha_{\sigma^*(f \circ g)} & & \\
\sum_{j=1}^s f_{\sigma(1)}^* G_{j1}(g_{\sigma(1)}) \otimes \cdots \otimes f_{\sigma(n)}^* G_{jn}(g_{\sigma(n)}) & &
\end{array}$$

Here

$$\begin{aligned}
a &= \left(\sum_{i=1}^r \tau_{\sigma, f_1^* F_{i\sigma^{-1}(1)}, \dots, f_n^* F_{i\sigma^{-1}(n)}} \right) (\underline{g}) \\
b &= \left(\sum_{i=1}^r \tau_{\sigma, f_1^* G_{i\sigma^{-1}(1)}, \dots, f_n^* G_{i\sigma^{-1}(n)}}^{-1} \right) (\underline{g}).
\end{aligned}$$

The naturality of $\Psi(\sigma, \underline{f})\alpha$ follows by the naturality of α and the maps

$$\tau_{\sigma, f_1^* F_{i1}, \dots, f_n^* F_{in}} \quad \text{and} \quad \tau_{\sigma, f_1^* G_{j1}, \dots, f_n^* G_{jn}}.$$

We need to check that $\Psi(\sigma, \underline{f})\alpha$ is base point preserving. To do so, note that $\Lambda((\Psi(\sigma, \underline{f})\alpha)_{\underline{g}}) = \tau_{\sigma, w_1, \dots, w_n}^{-1} \circ \tau_{\sigma, w_1, \dots, w_n} = \text{id}$, and thus $(\Psi(\sigma, \underline{f})\alpha)_{\underline{g}}$ is a morphism in the category $\mathcal{D}_{\otimes w_i}$. This defines $\Psi(q, \underline{f}) : \Psi(\underline{u}) \rightarrow \Psi(\underline{v})$ in the case that $q : \underline{n} \rightarrow \underline{m}$ is an isomorphism. Let us define $\Psi(q, \underline{f})$ for the case where $q : \underline{n} \rightarrow \underline{n+1}$ is the injective map that misses the value $n+1$, $\underline{v} = (u_1, \dots, u_n, 1)$ and each $\underline{f} = \text{id}$. Thus in this case $u'_i = u_i$ for $1 \leq i \leq n$ and $u'_{n+1} = 1$. Let us fix $I : \mathcal{C}/1 \rightarrow \mathcal{D}$ a morphism of fibered categories such that $I(\text{id}) = 1$. Then for an object

$$F = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in}$$

of $\Psi(\underline{u})$, we define

$$\Psi(q, \underline{f})F = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in} \otimes I.$$

This way defined we see that $\Psi(q, \underline{f})F$ is an object of $\Psi(\underline{v})$. On the other hand, if $\alpha : F \rightarrow G$ is a morphism in the category $\Psi(\underline{u})$, where

$$F = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in} \quad \text{and} \quad G = \sum_{j=1}^s G_{j1} \otimes \cdots \otimes G_{jn}.$$

then we define $\Psi(q, \underline{f})\alpha$ to be the morphism defined in the following diagram

$$\begin{array}{ccc} \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in} \otimes I & \xrightarrow{\Psi(q, \underline{f})\alpha} & G = \sum_{j=1}^s G_{j1} \otimes \cdots \otimes G_{jn} \otimes I \\ \cong \downarrow & & \downarrow \cong \\ (\sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in}) \otimes I & \xrightarrow{\alpha \otimes \text{id}} & (\sum_{j=1}^s G_{j1} \otimes \cdots \otimes G_{jn}) \otimes I, \end{array}$$

where the vertical arrows are obtained by application of the distributivity maps in \mathcal{D} . It is easy to see that this way defined $\Psi(q, \underline{f})\alpha$ is a base point preserving natural transformations. This defines $\Psi(q, \underline{f})$ in this case. Note that any morphism in \mathcal{A} can be factorize into a composition of the previous form, hence this way we define $\Psi(q, \underline{f})$ for every morphism (q, \underline{f}) in \mathcal{A} . By a direct inspection, we see that $\Psi(\sigma, \underline{f})$ is a lax_{*} map, in fact it is a strict map. We conclude then that $\Psi : \mathcal{A} \rightarrow \mathbb{P}$ is a functor. Thus we have proved the following theorem.

Theorem 15 *To every fibered bipermutative category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ we can associate a functor $\Psi : \mathcal{A} \rightarrow \mathbb{P}$. The functor determines and is determined by $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ up to canonical isomorphism.*

In Theorem 26, we will show that the functor $\Psi : \mathcal{A} \rightarrow \mathbb{P}$ has special properties arising from the multiplicative structure of the fibered bipermutative category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$. The functor Ψ motivates the study in general of functors out of a permutative category to the multicategory of permutative categories or in general to a multicategory. We show in the next section that we can give such functors the structure of a multicategory.

4 Multicategories and general construction

In this section we review multicategories. Roughly speaking a multicategory or colored operad, is a generalization of both operads and symmetric monoidal categories. Our main goal in this section is to show that given a permutative category $(\mathcal{E}, \otimes, 1)$ and a multicategory \mathcal{M} , we can give the structure of a multicategory to the category of functors $\mathcal{E} \rightarrow \mathcal{M}$.

Definition 16 *A multicategory \mathcal{M} consists of the following:*

- (1) *A collection of objects which we usually denote $\mathcal{O}_{\mathcal{M}}$.*
- (2) *For $k \geq 0$ and any $(k + 1)$ -tuple of objects a_1, \dots, a_k and b , a set*

$$\mathcal{M}(a_1, \dots, a_k; b),$$

called the set of “colored” k -morphisms.

(3) A right action of Σ_k on the collection of all k -morphisms, where for $\sigma \in \Sigma_k$

$$\sigma^* : \mathcal{M}(a_1, \dots, a_k; b) \rightarrow \mathcal{M}(a_{\sigma(1)}, \dots, a_{\sigma(k)}, b).$$

(4) A 1-morphism $1_a \in \mathcal{M}(a; a)$ called the unit, for each object a of \mathcal{M} .

(5) A multiproduct function

$$\begin{aligned} \Gamma_{\mathcal{M}} : \mathcal{M}(b_1, \dots, b_n; c) \times \mathcal{M}(a_{11}, \dots, a_{1j_1}; b_1) \times \dots \times \mathcal{M}(a_{n1}, \dots, a_{nj_n}; b_n) \\ \rightarrow \mathcal{M}(a_{11}, \dots, a_{nj_n}; c) \end{aligned}$$

that satisfies properties Multi(1)-Multi(4) in (4, Definition 2.1) that generalize the properties of an operad as in (12, Definition 1.1).

As mentioned before a multicategory is a generalization of both operads and symmetric monoidal categories, to be more precise, an operad can be seen as a multicategory with only one object. Also, if $(\mathcal{E}, \oplus, 0)$ is a symmetric permutative category, we can see \mathcal{E} as a multicategory by considering as objects the objects of \mathcal{E} and for c_1, \dots, c_k, d objects of \mathcal{E} , the set of k -morphisms

$$\mathcal{E}(c_1, \dots, c_k; d) := \mathcal{E}(c_1 \oplus \dots \oplus c_k, d).$$

Here $c_1 \oplus \dots \oplus c_k$ means the iterated application of \oplus by inserting parenthesis in a consistent way.

In (4), Elmendorf and Mandell gave the category \mathbb{P} the structure of a multicategory. To describe the k -morphisms of this multicategory we need the following definition.

Definition 17 Let $\mathcal{C}_1, \dots, \mathcal{C}_k$ and \mathcal{D} be small permutative categories. A functor

$$f : \mathcal{C}_1 \times \dots \times \mathcal{C}_k \rightarrow \mathcal{D}$$

is said to be a k -linear map, if $f(c_1, \dots, c_k) = 0$ whenever $c_i = 0$ for some $1 \leq i \leq k$ and in addition, we have natural transformations δ^i , for $1 \leq i \leq k$, that are thought of as distributivity maps,

$$\delta_f^i = \delta^i : f(c_1, \dots, c_i, \dots, c_k) \oplus f(c_1, \dots, c'_i, \dots, c_k) \rightarrow f(c_1, \dots, c_i \oplus c'_i, \dots, c_k).$$

These transformations are such that $\delta^i = id$ whenever either c_i or c'_i is 0, or if any of the other c_j 's are 0. In addition, these natural transformations are subject to commutativity of a suitable collection of diagrams as in definition (4, 3.2).

If $\mathcal{C}_1, \dots, \mathcal{C}_k, \mathcal{D}$ are small permutative categories, then $\mathbb{P}(\mathcal{C}_1, \dots, \mathcal{C}_k, \mathcal{D})$, the set of k -morphisms in the multicategory \mathbb{P} , is precisely the set of all k -linear maps

$$f : \mathcal{C}_1 \times \dots \times \mathcal{C}_k \rightarrow \mathcal{D}.$$

The k -morphism set $\mathbb{P}(\mathcal{C}_1, \dots, \mathcal{C}_k, \mathcal{D})$ forms a category. If

$$f, g : \mathcal{C}_1 \times \dots \times \mathcal{C}_k \rightarrow \mathcal{D}$$

are two k -linear maps, then a morphism $\alpha : f \rightarrow g$ is a natural transformation such that $\alpha(c_1, \dots, c_k)$ is the identity map whenever $c_i = 0$ for some i and also the following diagram is commutative for each $1 \leq i \leq k$

$$\begin{array}{ccc} f(c_1, \dots, c_i, \dots, c_k) \oplus f(c_1, \dots, c'_i, \dots, c_k) & \xrightarrow{\delta_f^i} & f(c_1, \dots, c_i \oplus c'_i, \dots, c_k) \\ \alpha \oplus \alpha \downarrow & & \downarrow \alpha \\ g(c_1, \dots, c_i, \dots, c_k) \oplus g(c_1, \dots, c'_i, \dots, c_k) & \xrightarrow{\delta_g^i} & g(c_1, \dots, c_i \oplus c'_i, \dots, c_k). \end{array}$$

Throughout this paper we will be dealing with functors and multifunctors in a setup where the two concepts make sense (and are different). To avoid confusion we will start by setting up the notation that will be used through this paper. In this section \mathcal{M} and \mathcal{N} will denote general multicategories enriched over \mathbf{Cat} and $(\mathcal{E}, \oplus, 0)$ a general permutative category. Note that as explained before we can see \mathcal{E} as a multicategory and also we can see \mathcal{M} as a category by consider the objects and 1-morphisms and forgetting the rest of the data.

Notation:

- We will denote by $\mathbf{F}(\mathcal{E}, \mathcal{M})$ or $\mathcal{M}^{\mathcal{E}}$ the category whose objects are functors $\mathcal{E} \rightarrow \mathcal{M}$ and the morphisms are the natural transformation between them.
- Also, we will denote by $\mathbf{M}(\mathcal{E}, \mathcal{M})$ the category whose objects are multifunctors $\mathcal{E} \rightarrow \mathcal{M}$ and the morphisms the natural transformations preserving the multiproduct.
- Finally, by $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \mathcal{N})$ we mean the category whose objects are assignments $F : \mathcal{M} \times \mathcal{E} \rightarrow \mathcal{N}$ that are multifunctors in the first component and are functors in the second component and the morphisms are the transformations preserving the multi-structure in the first component and are natural transformations in the second component.

Remark 18 *As categories, $\mathbf{M}(\mathcal{M}, \mathcal{N}^{\mathcal{E}})$ and $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \mathcal{N})$ are naturally isomorphic. In what follows, we will identify those two categories without further comment.*

We will denote by $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{O}_{\mathcal{M}}$ the object sets of \mathcal{E} and \mathcal{M} respectively. We are interested in studying the functors

$$\mathcal{E} \rightarrow \mathcal{M}$$

and want to construct a multicategory $\mathcal{M}^{\mathcal{E}}$, whose objects are precisely those functors. The goal of this section is then to prove the following theorem.

Theorem 19 *There is a multicategory $\mathcal{M}^\mathcal{E}$ whose objects are the functors $\mathcal{E} \rightarrow \mathcal{M}$. This multicategory is enriched over \mathbf{Cat} if \mathcal{M} is enriched over \mathbf{Cat} .*

Before proving this we need some remarks and definitions. First of all, note that given a functor $G : \mathcal{E} \rightarrow \mathcal{M}$ and any $k \geq 0$, we can see G as a functor $G : \mathcal{E}^k \rightarrow \mathcal{M}$ by considering $G(c_1, \dots, c_k) = G(c_1 \otimes \dots \otimes c_k)$. With this in mind we can give the following definition.

Definition 20 *Suppose that $F_1, \dots, F_k, G : \mathcal{E} \rightarrow \mathcal{M}$ are functors. A k -linear natural transformation from F_1, \dots, F_k to G , is an assignment that for each k -tuple $c_1, \dots, c_k \in \mathcal{O}_\mathcal{E}$, corresponds a k -morphism*

$$\phi_{c_1, \dots, c_k} \in \mathcal{M}(F_1(c_1), \dots, F_k(c_k), G(\otimes_{i=1}^k c_i))$$

satisfying the naturality condition described below.

The naturality condition that we require is the following: suppose that $f_i : c_i \rightarrow c'_i$ is a morphisms in \mathcal{E} for $1 \leq i \leq k$, then we have 1-morphisms $F_i(f_i) : F(c_i) \rightarrow F(c'_i)$ for $1 \leq i \leq k$ and $G(\otimes_{i=1}^k f_i) : G(\otimes_{i=1}^k c_i) \rightarrow G(\otimes_{i=1}^k c'_i)$. The multiproduct $\Gamma_{\mathcal{M}}$ of \mathcal{M} gives maps

$$\begin{aligned} \Gamma_{\mathcal{M}} : & \mathcal{M}(G(\otimes_{i=1}^k c_i); G(\otimes_{i=1}^k c'_i)) \times \mathcal{M}(F_1(c_1), \dots, F_k(c_k); G(\otimes_{i=1}^k c_i)) \\ & \rightarrow \mathcal{M}(F_1(c_1), \dots, F_k(c_k); G(\otimes_{i=1}^k c'_i)), \\ \Gamma_{\mathcal{M}} : & \mathcal{M}(F_1(c'_1), \dots, F_k(c'_k); G(\otimes_{i=1}^k c'_i)) \times \prod_{i=1}^k \mathcal{M}(F_i(c_i); F_i(c'_i)) \\ & \rightarrow \mathcal{M}(F_1(c_1), \dots, F_k(c_k); G(\otimes_{i=1}^k c'_i)). \end{aligned}$$

We require that

$$\Gamma_{\mathcal{M}}(G(\otimes_{i=1}^k f_i); \phi_{c_1, \dots, c_k}) = \Gamma_{\mathcal{M}}(\phi_{c'_1, \dots, c'_k}; F_1(f_1), \dots, F_k(f_k)).$$

Remark 21 *Note that if we take $k = 1$ in the previous definition, then a 1-linear natural transformation $\phi : F \rightarrow G$, is an assignment that for $c \in \mathcal{O}_\mathcal{E}$, a 1-linear map $\phi_c : F(c) \rightarrow G(c)$ and for each morphism $f : c \rightarrow c'$ in \mathcal{E}*

$$\Gamma_{\mathcal{M}}(G(f); \phi_c) = \Gamma_{\mathcal{M}}(\phi_{c'}; F(f))$$

This means that $\phi : F \rightarrow G$ is a natural transformation such that ϕ_c is 1-linear for each object c of \mathcal{E} .

Proof of Theorem 19. To prove Theorem 19 we need to define a class of objects, a set of k -morphisms for $k \geq 0$, a right action of Σ_k on the set of k -morphisms, a unit 1_F for every object F of $\mathcal{M}^\mathcal{E}$ and a multiproduct such that properties Multi(1)-Multi(4) of (4, Definition 2.1) are satisfied.

The objects of $\mathcal{M}^\mathcal{E}$ are the functors $F : \mathcal{E} \rightarrow \mathcal{M}$. To define the k -morphisms in $\mathcal{M}^\mathcal{E}$, suppose that $F_1, \dots, F_k, G : \mathcal{E} \rightarrow \mathcal{M}$ are objects in $\mathcal{M}^\mathcal{E}$. Then a

k -morphism in $\mathcal{M}^\mathcal{E}(F_1, \dots, F_k; G)$ is a k -linear natural transformation from F_1, \dots, F_k to G . When $k = 0$, then a 0-morphism in $\mathcal{M}^\mathcal{E}(, G)$ is a 0-morphism in $\mathcal{M}(, F(1))$, where 1 is the coherent unit of \mathcal{E} .

The multicategory $\mathcal{M}^\mathcal{E}$ will be an enriched category over \mathbf{Cat} whenever \mathcal{M} is enriched over \mathbf{Cat} . Thus for objects F_1, \dots, F_k, G in $\mathcal{M}^\mathcal{E}$, $\mathcal{M}^\mathcal{E}(F_1, \dots, F_k, G)$ is the category whose objects are the k -linear natural transformations from F_1, \dots, F_k to G . If ϕ, ψ are two such transformations, then a morphism $f : \phi \rightarrow \psi$ in $\mathcal{M}^\mathcal{E}(F_1, \dots, F_k, G)$, is an assignment, for objects c_i , $1 \leq i \leq k$, a morphism $f_{c_1, \dots, c_k} : \phi_{c_1, \dots, c_k} \rightarrow \psi_{c_1, \dots, c_k}$ in the category $\mathcal{M}^\mathcal{E}(F_1(c_1), \dots, F_k(c_k), G(\otimes_{i=1}^k c_i))$. Let's define the multiproduct on $\mathcal{M}^\mathcal{E}$. Suppose that $\phi^i \in \mathcal{M}^\mathcal{E}(F_{i1}, \dots, F_{ij_i}; G_i)$ is a j_i -morphism for $1 \leq i \leq k$ and that $\psi \in \mathcal{M}^\mathcal{E}(G_1, \dots, G_k; H)$ is an k -morphism. We want to define $\Gamma_{\mathcal{M}^\mathcal{E}}(\psi; \phi^1, \dots, \phi^k) \in \mathcal{M}^\mathcal{E}(F_{11}, \dots, F_{kj_k}; H)$, a $(j_1 + \dots + j_k)$ -morphism. To do so, take $c_{11}, \dots, c_{kj_k} \in \mathcal{O}_\mathcal{E}$ and define

$$\Gamma_{\mathcal{M}^\mathcal{E}}(\psi; \phi^1, \dots, \phi^k)_{c_{11}, \dots, c_{kj_k}} = \Gamma_{\mathcal{M}}(\psi_{\otimes_{r=1}^{j_1} c_{1r}, \dots, \otimes_{r=1}^{j_k} c_{kr}}; \phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{k1}, \dots, c_{kj_k}}^k).$$

This definition readily extends to morphisms in the case that \mathcal{M} is enriched over \mathbf{Cat} . We need to show that $\Gamma_{\mathcal{M}^\mathcal{E}}(\psi; \phi^1, \dots, \phi^k)$ satisfies the naturality condition. Since \mathcal{M} is a multicategory, then by definition we have that

$$\Gamma_{\mathcal{M}}(\psi_{\otimes_{r=1}^{j_1} c_{1r}, \dots, \otimes_{r=1}^{j_k} c_{kr}}; \phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{k1}, \dots, c_{kj_k}}^k)$$

is a $(j_1 + \dots + j_k)$ -morphism. Suppose then that we have morphisms $f_{rs} : c_{rs} \rightarrow c'_{rs}$ in \mathcal{E} , for $1 \leq r \leq k$ and $1 \leq s \leq j_r$. The multiproduct of \mathcal{M} gives us the maps

$$\begin{aligned} \Gamma_{\mathcal{M}} : & \mathcal{M}(H(\otimes_{r,s} c_{rs}); H(\otimes_{r,s} c'_{rs})) \times \mathcal{M}(F_{11}(c_{11}), \dots, F_{kj_k}(c_{kj_k}); H(\otimes_{r,s} c_{rs})) \\ & \rightarrow \mathcal{M}(F_{11}(c_{11}), \dots, F_{kj_k}(c_{kj_k}); H(\otimes_{r,s} c'_{rs})) \\ \Gamma_{\mathcal{M}} : & \mathcal{M}(F_{11}(c'_{11}), \dots, F_{kj_k}(c'_{kj_k}), H(\otimes_{r,s} c'_{rs})) \times \prod_{r,s} \mathcal{M}(F_{rs}(c_{rs}), F_{rs}(c'_{rs})) \\ & \rightarrow \mathcal{M}(F_{11}(c_{11}), \dots, F_{kj_k}(c_{kj_k}), H(\otimes_{r,s} c'_{rs})). \end{aligned}$$

We need to check that

$$\begin{aligned} & \Gamma_{\mathcal{M}}(H(\otimes_{r,s} f_{rs}); \Gamma_{\mathcal{M}^\mathcal{E}}(\psi; \phi^1, \dots, \phi^k)_{c_{11}, \dots, c_{kj_k}}) \\ & = \Gamma_{\mathcal{M}}(\Gamma_{\mathcal{M}^\mathcal{E}}(\psi; \phi^1, \dots, \phi^k)_{c'_{11}, \dots, c'_{kj_k}}; F_{11}(f_{11}), \dots, F_{kj_k}(f_{kj_k})). \end{aligned}$$

On the one hand by definition and property Multi(1) for \mathcal{M} we have

$$\begin{aligned} & \Gamma_{\mathcal{M}}(H(\otimes_{r,s} f_{rs}); \Gamma_{\mathcal{M}^\mathcal{E}}(\psi; \phi^1, \dots, \phi^k)_{c_{11}, \dots, c_{kj_k}}) \\ & = \Gamma_{\mathcal{M}}(H(\otimes_{r,s} f_{rs}); \Gamma_{\mathcal{M}}(\psi_{\otimes_{r=1}^{j_1} c_{1r}, \dots, \otimes_{r=1}^{j_k} c_{kr}}; \phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{k1}, \dots, c_{kj_k}}^k)) \\ & = \Gamma_{\mathcal{M}}(\Gamma_{\mathcal{M}}(H(\otimes_{r,s} f_{rs}); \psi_{\otimes_{r=1}^{j_1} c_{1r}, \dots, \otimes_{r=1}^{j_k} c_{kr}}); \phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{k1}, \dots, c_{kj_k}}^k) \end{aligned}$$

but as ψ is an k -morphism in $\mathcal{M}^\mathcal{E}$, we have that

$$\begin{aligned} & \Gamma_{\mathcal{M}}(H(\otimes_{r,s} f_{rs}); \psi_{\otimes_{r=1}^{j_1} c_{1r}, \dots, \otimes_{r=1}^{j_k} c_{kr}}) = \\ & \Gamma_{\mathcal{M}}(\psi_{\otimes_{r=1}^{j_1} c'_{1r}, \dots, \otimes_{r=1}^{j_k} c'_{kr}}; G_1(\otimes_s f_{1s}), \dots, G_k(\otimes_s f_{ks})) \end{aligned}$$

hence

$$\begin{aligned} & \Gamma_{\mathcal{M}}(H(\otimes_{r,s} f_{rs}); \Gamma_{\mathcal{M}^\mathcal{E}}(\psi; \phi^1, \dots, \phi^k)_{c_{11}, \dots, c_{kj_k}}) = \tag{1} \\ & \Gamma_{\mathcal{M}}(\Gamma_{\mathcal{M}}(\psi_{\otimes_{r=1}^{j_1} c'_{1r}, \dots, \otimes_{r=1}^{j_k} c'_{kr}}; G_1(\otimes_s f_{1s}), \dots, G_k(\otimes_s f_{ks})); \phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{k1}, \dots, c_{kj_k}}^k). \tag{2} \end{aligned}$$

On the other hand, again using the definition and property Multi(1) for \mathcal{M} we have

$$\begin{aligned} & \Gamma_{\mathcal{M}}(\Gamma_{\mathcal{M}^\mathcal{E}}(\psi; \phi^1, \dots, \phi^k)_{c'_{11}, \dots, c'_{kj_k}}; F_{11}(f_{11}), \dots, F_{kj_k}(f_{kj_k})) \\ & = \Gamma_{\mathcal{M}}(\Gamma_{\mathcal{M}}(\psi_{\otimes_{r=1}^{j_1} c'_{1r}, \dots, \otimes_{r=1}^{j_k} c'_{kr}}; \phi_{c'_{11}, \dots, c'_{1j_1}}^1, \dots, \phi_{c'_{k1}, \dots, c'_{kj_k}}^k); F_{11}(f_{11}), \dots, F_{kj_k}(f_{kj_k})) \\ & = \Gamma_{\mathcal{M}}(\psi_{\otimes_{r=1}^{j_1} c'_{1r}, \dots, \otimes_{r=1}^{j_k} c'_{kr}}; \alpha_1, \dots, \alpha_k), \end{aligned}$$

where

$$\alpha_i = \Gamma_{\mathcal{M}}(\phi_{c'_{i1}, \dots, c'_{ij_i}}^i; F_{i1}(f_{i1}), \dots, F_{ij_i}(f_{ij_i})).$$

But as each ϕ^i is a j_i -morphism, by definition we get that

$$\alpha_i = \Gamma_{\mathcal{M}}(G(\otimes_s f_{is}); \phi_{c_{i1}, \dots, c_{ij_i}}^i) = \Gamma_{\mathcal{M}}(\phi_{c'_{i1}, \dots, c'_{ij_i}}^i; F_{i1}(f_{i1}), \dots, F_{ij_i}(f_{ij_i})),$$

therefore

$$\begin{aligned} & \Gamma_{\mathcal{M}}(\Gamma_{\mathcal{M}^\mathcal{E}}(\psi; \phi^1, \dots, \phi^k)_{c'_{11}, \dots, c'_{kj_k}}; F_{11}(f_{11}), \dots, F_{kj_k}(f_{kj_k})) = \tag{3} \\ & \Gamma_{\mathcal{M}}(\Gamma_{\mathcal{M}}(\psi_{\otimes_{r=1}^{j_1} c'_{1r}, \dots, \otimes_{r=1}^{j_k} c'_{kr}}; G_1(\otimes_s f_{1s}), \dots, G_k(\otimes_s f_{ks})); \phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{k1}, \dots, c_{kj_k}}^k) \tag{4} \end{aligned}$$

By (1) and (3) we see that $\Gamma_{\mathcal{M}^\mathcal{E}}(\psi; \phi^1, \dots, \phi^k)$ also satisfies the naturality condition.

We define now a right action of Σ_k on the set of all k -morphisms. Suppose then that $\phi \in \mathcal{M}^\mathcal{E}(F_1, \dots, F_k; G)$ is a k -morphism in $\mathcal{M}^\mathcal{E}$ and $\sigma \in \Sigma_k$. Define $\sigma^* \phi \in \mathcal{M}^\mathcal{E}(F_{\sigma(1)}, \dots, F_{\sigma(k)}, G)$ as follows: for objects c_1, \dots, c_k of \mathcal{E} , define

$$\sigma^* \phi_{c_{\sigma(1)}, \dots, c_{\sigma(k)}} = \sigma^* \Gamma(G(\tau_{\sigma, c_1, \dots, c_k}); \phi_{c_1, \dots, c_k}).$$

Here

$$\tau_{\sigma, c_1, \dots, c_k} : c_1 \otimes \dots \otimes c_k \rightarrow c_{\sigma(1)} \otimes \dots \otimes c_{\sigma(k)},$$

is the natural isomorphism in \mathcal{E} and

$$\begin{aligned} \sigma^* : \mathcal{M}(F_1(c_1), \dots, F_k(c_k), G(\otimes_{i=1}^k c_{\sigma(i)})) & \rightarrow \\ \mathcal{M}(F_{\sigma(1)}(c_{\sigma(1)}), \dots, F_{\sigma(k)}(c_{\sigma(k)}), G(\otimes_{i=1}^k c_{\sigma(i)})) & \end{aligned}$$

is the action of Σ_k on the k -morphisms of the multicategory \mathcal{M} . By a similar argument as in the definition of the multiproduct, it follows that this way defined, $\sigma^*\phi$ satisfies the naturality condition and thus $\sigma^*\phi$ is well defined. This definition extends easily to morphisms in the category of k -morphisms when \mathcal{M} is enriched over \mathbf{Cat} . This indeed defines an action as Σ_k acts on the right on the set of k -morphisms of \mathcal{M} . If $F : \mathcal{E} \rightarrow \mathcal{M}$ is an object of $\mathcal{M}^\mathcal{E}$, then the identity $1_F \in \mathcal{M}^\mathcal{E}(F; F)$ is the 1-linear natural transformation 1_F , such that for each object c of \mathcal{E} , $(1_F)_c = 1_{F(c)} \in \mathcal{M}(F(c); F(c))$. 1_F clearly satisfies the naturality condition and thus it defines a 1-morphism in $\mathcal{M}^\mathcal{E}(F; F)$ that is clearly a unit.

Let's check now that this data defines a multicategory; that is, we need to check the conditions Multi(1)-Multi(4) of (4, Definition 2.1). Conditions Multi(1) and Multi(2) are satisfied pointwise as \mathcal{M} is a multicategory and thus these conditions hold for $\mathcal{M}^\mathcal{E}$. Let's check that condition Multi(3) is satisfied. To do so, take F_{ir}, G_i and H objects in $\mathcal{M}^\mathcal{E}$ for $1 \leq i \leq k$ and $1 \leq r \leq j_i$. Also take $\sigma \in \Sigma_k$. We need to show the commutativity of the following diagram

$$\begin{array}{ccc}
\mathcal{M}^\mathcal{E}(G_1, \dots, G_k; H) \times \prod_{i=1}^k \mathcal{M}^\mathcal{E}(F_{i1}, \dots, F_{ij_i}; G_i) & & \\
\downarrow \sigma^* \times \sigma^{-1} & \searrow \Gamma & \\
& & \mathcal{M}^\mathcal{E}(F_{11}, \dots, F_{kj_k}; H) \\
& & \downarrow (\sigma_{\langle j_{\sigma(1)}, \dots, j_{\sigma(k)} \rangle})^* \\
\mathcal{M}^\mathcal{E}(G_{\sigma(1)}, \dots, G_{\sigma(k)}; H) \times \prod_{i=1}^k \mathcal{M}^\mathcal{E}(F_{\sigma(i)1}, \dots, F_{\sigma(i)j_{\sigma(i)}}; G_{\sigma(i)}) & \searrow \Gamma & \\
& & \mathcal{M}^\mathcal{E}(F_{\sigma(1)1}, \dots, F_{\sigma(k)j_{\sigma(k)}}; H).
\end{array}$$

To show the commutativity of this diagram, take objects c_{11}, \dots, c_{kj_k} of \mathcal{E} , $\phi^i \in \mathcal{M}^\mathcal{E}(F_{i1}, \dots, F_{ij_i}; G_i)$ for $1 \leq i \leq k$ and $\psi \in \mathcal{M}^\mathcal{E}(G_1, \dots, G_k; H)$ an k -morphism. Then by definition we have that

$$\begin{aligned}
& ((\sigma_{\langle j_{\sigma(1)}, \dots, j_{\sigma(k)} \rangle})^* \Gamma(\psi, \phi^1, \dots, \phi^k))_{c_{\sigma(1)1}, \dots, c_{\sigma(k)j_{\sigma(k)}}} = \\
& (\sigma_{\langle j_{\sigma(1)}, \dots, j_{\sigma(k)} \rangle})^* \Gamma(H(\tau_{\sigma, c_{11}, \dots, c_{kj_k}}); \Gamma(\psi; \phi^1, \dots, \phi^k)_{c_{11}, \dots, c_{kj_k}}) = \\
& (\sigma_{\langle j_{\sigma(1)}, \dots, j_{\sigma(k)} \rangle})^* \Gamma(H(\tau_{\sigma, c_{11}, \dots, c_{kj_k}}); \Gamma(\psi_{\otimes_{i=1}^{j_1} c_{1i}, \dots, \otimes_{i=1}^{j_k} c_{ki}}; \phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{k1}, \dots, c_{kj_k}}^1})).
\end{aligned}$$

On the other hand

$$\begin{aligned}
& \Gamma((\sigma^* \times \sigma^{-1})(\psi; \phi^1, \dots, \phi^k))_{c_{\sigma(1)1}, \dots, c_{\sigma(k)j_{\sigma(k)}}} = \\
& \Gamma(\sigma^* \psi; \phi^{\sigma(1)}, \dots, \phi^{\sigma(k)})_{c_{\sigma(1)1}, \dots, c_{\sigma(k)j_{\sigma(k)}}} = \\
& \Gamma(\sigma^* \psi_{\otimes_{i=1}^{j_{\sigma(1)}} c_{\sigma(1)i}, \dots, \otimes_{i=1}^{j_{\sigma(k)}} c_{\sigma(k)i}}; \phi_{c_{\sigma(1)1}, \dots, c_{\sigma(1)j_{\sigma(1)}}}^{\sigma(1)}, \dots, \phi_{c_{\sigma(k)1}, \dots, c_{\sigma(k)j_{\sigma(k)}}}^{\sigma(k)}) = \\
& \Gamma(\sigma^* \Gamma(H(\tau_{\sigma, c_{11}, \dots, c_{kj_k}}); \psi_{\otimes_{i=1}^{j_1} c_{1i}, \dots, \otimes_{i=1}^{j_k} c_{ki}}); \phi_{c_{\sigma(1)1}, \dots, c_{\sigma(1)j_{\sigma(1)}}}^{\sigma(1)}, \dots, \phi_{c_{\sigma(k)1}, \dots, c_{\sigma(k)j_{\sigma(k)}}}^{\sigma(k)}).
\end{aligned}$$

Since \mathcal{M} is a multicategory, by property Multi(3) we get

$$\begin{aligned} & \Gamma(\sigma^* \Gamma(H(\tau_{\sigma, c_{11}}, \dots, c_{kj_k}); \psi_{\otimes_{c_{1i}, \dots, \otimes_{c_{ki}}}); \phi_{c_{\sigma(1)1}, \dots, c_{\sigma(1)j_{\sigma(1)}}}^{\sigma(1)}, \dots, \phi_{c_{\sigma(k)1}, \dots, c_{\sigma(k)j_{\sigma(k)}}}^{\sigma(k)})) \\ &= \Gamma((\sigma^* \times \sigma^{-1})(\Gamma(H(\tau_{\sigma, c_{11}}, \dots, c_{kj_k}); \psi_{\otimes_{c_{1i}, \dots, \otimes_{c_{ki}}}); \phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{k1}, \dots, c_{kj_k}}^1))) \\ &= (\sigma_{\langle j_{\sigma(1)}, \dots, j_{\sigma(k)} \rangle})^* \Gamma(\Gamma(H(\tau_{\sigma, c_{11}}, \dots, c_{kj_k}); \psi_{\otimes_{c_{1i}, \dots, \otimes_{c_{ki}}}); \phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{k1}, \dots, c_{kj_k}}^1))), \end{aligned}$$

but again using the fact that \mathcal{M} is a multicategory, by property Multi(1) we get that

$$\begin{aligned} & \Gamma(\Gamma(H(\tau_{\sigma, c_{11}}, \dots, c_{kj_k}); \psi_{\otimes_{i=1}^{j_1} c_{1i}, \dots, \otimes_{i=1}^{j_k} c_{ki}}); \phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{k1}, \dots, c_{kj_k}}^1)) \\ &= \Gamma(H(\tau_{\sigma, c_{11}, \dots, c_{kj_k}}); \Gamma(\psi_{\otimes_{i=1}^{j_1} c_{1i}, \dots, \otimes_{i=1}^{j_k} c_{ki}}); \phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{k1}, \dots, c_{kj_k}}^1)). \end{aligned}$$

Thus diagram Multi(3) commutes. In the case that \mathcal{M} is enriched over \mathbf{Cat} , a similar argument shows that the respective diagram commutes on morphisms. The commutativity of diagram Multi(4) is similar. This proves Theorem 19. \square

Example: Our main application of the previous theorem is the case where $\mathcal{M} = \mathbb{P}$. Explicitly, the multicategory $\mathbb{P}^{\mathcal{E}}$, has objects the functors

$$\mathcal{E} \rightarrow \mathbb{P}.$$

If F_1, \dots, F_k and G are objects in $\mathbb{P}^{\mathcal{E}}$, then the k -morphism set, $\mathbb{P}^{\mathcal{E}}(F_1, \dots, F_k; G)$ forms a category, its objects are the k -linear natural maps; that is, the assignments ϕ that for each $c_1, \dots, c_k \in \mathcal{O}_{\mathcal{E}}$ correspond a k -linear map

$$\phi_{c_1, \dots, c_k} : F_1(c_1) \times \dots \times F_k(c_k) \rightarrow G(c_1 \otimes \dots \otimes c_k)$$

such that ϕ satisfies the naturality condition as described before. If ϕ and ψ are two objects in $\mathbb{P}^{\mathcal{E}}(F_1, \dots, F_k; G)$, then a morphism $\alpha : \phi \rightarrow \psi$ is a morphism of natural k -linear transformations; that is, for c_1, \dots, c_k

$$\alpha_{c_1, \dots, c_k} : \phi_{c_1, \dots, c_k} \rightarrow \psi_{c_1, \dots, c_k}$$

is a natural map such that the following diagram is commutative

$$\begin{array}{ccc} \phi_{c_1, \dots, c_k}(x_1, \dots, x_i, \dots, x_k) \oplus \phi_{c_1, \dots, c_k}(x_1, \dots, x'_i, \dots, x_k) & \xrightarrow{\delta_{\phi_{c_1, \dots, c_k}}^i} & \phi_{c_1, \dots, c_k}(x_1, \dots, x_i \oplus x'_i, \dots, x_k) \\ \downarrow \alpha_{c_1, \dots, c_k} \oplus \alpha_{c_1, \dots, c_k} & & \downarrow \alpha_{c_1, \dots, c_k} \\ \psi_{c_1, \dots, c_k}(x_1, \dots, x_i, \dots, x_k) \oplus \psi_{c_1, \dots, c_k}(x_1, \dots, x'_i, \dots, x_k) & \xrightarrow{\delta_{\psi_{c_1, \dots, c_k}}^i} & \psi_{c_1, \dots, c_k}(x_1, \dots, x_i \oplus x'_i, \dots, x_k). \end{array}$$

Here, $\delta_{\phi_{c_1, \dots, c_k}}^i$ and $\delta_{\psi_{c_1, \dots, c_k}}^i$ are the structural maps of the k -linear maps ϕ_{c_1, \dots, c_k} and ψ_{c_1, \dots, c_k} .

The action of Σ_k on k -morphisms, is defined in the following way. If $\sigma \in \Sigma_k$ and

$\phi \in \mathbb{P}^{\mathcal{E}}(F_1, \dots, F_k, G)$. For $c_1, \dots, c_k \in \mathcal{O}_C$, define $\sigma^* \phi_{c_{\sigma(1)}, \dots, c_{\sigma(k)}}$ is the k -linear map making the following diagram commutative

$$\begin{array}{ccc} \prod_i^k F_{\sigma(i)}(c_{\sigma(i)}) & \xrightarrow{\sigma^* \phi_{c_{\sigma(1)}, \dots, c_{\sigma(k)}}} & G(\otimes_{i=1}^k c_{\sigma(i)}) \\ \sigma^{-1} \downarrow & & \uparrow G(\tau_{\sigma, c_1, \dots, c_k}) \\ \prod_i^k F_i(c_i) & \xrightarrow{\phi_{c_1, \dots, c_k}} & G(\otimes_{i=1}^k c_i) \end{array}$$

Here $\tau_{\sigma, c_1, \dots, c_k}$ is the coherent isomorphism $c_1 \otimes \dots \otimes c_k \xrightarrow{\approx} c_{\sigma(1)} \otimes \dots \otimes c_{\sigma(k)}$ in \mathcal{E} obtained by iterated applications of γ . On the other hand, the multi-product in $\mathbb{P}^{\mathcal{E}}$ is defined as follows: suppose $\phi^i \in \mathbb{P}^{\mathcal{E}}(F_{i1}, \dots, F_{ij_i}; G_i)$, $\psi \in \mathbb{P}^{\mathcal{E}}(G_1, \dots, G_n; H)$. For any objects c_{is} of \mathcal{E} , for $1 \leq i \leq n$, $1 \leq s \leq j_i$, define

$$\Gamma(\psi, \phi^1, \dots, \phi^n)_{c_{11}, \dots, c_{nj_n}} = \psi_{c_{11} \otimes \dots \otimes c_{1j_1}, \dots, c_{n1} \otimes \dots \otimes c_{nj_n}}(\phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{n1}, \dots, c_{nj_n}}^n).$$

Suppose now that \mathcal{M}_1 and \mathcal{M}_2 are two multicategories and that $\mathcal{F} : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a multifunctor. Then \mathcal{F} induces a functor $\mathcal{F}_* : \mathcal{M}_1^{\mathcal{E}} \rightarrow \mathcal{M}_2^{\mathcal{E}}$ defined by composition with \mathcal{F} . This functor preserves the multicategory structure; that is, \mathcal{F}_* is a multifunctor. In the case where that both \mathcal{M}_1 and \mathcal{M}_2 are enriched over **Cat** and that \mathcal{F} is an enriched multifunctor, then \mathcal{F}_* is also enriched. We prove this in the following theorem.

Theorem 22 *The functor \mathcal{F}_* is a multifunctor that is enriched if \mathcal{M}_1 and \mathcal{M}_2 are enriched over **Cat** and \mathcal{F} is an enriched multifunctor.*

Proof: Suppose that $F_1, \dots, F_k, G : \mathcal{E} \rightarrow \mathcal{M}_1$ are objects in $\mathcal{M}_1^{\mathcal{E}}$, then $\mathcal{F}_*(F_1) = \mathcal{F} \circ F_1, \dots, \mathcal{F}_*(G) = \mathcal{F} \circ G : \mathcal{E} \rightarrow \mathcal{M}_2$. Take $\phi \in \mathcal{M}_1^{\mathcal{E}}(F_1, \dots, F_k, G)$. Thus given objects c_1, \dots, c_k in \mathcal{E} , by definition $\phi_{c_1, \dots, c_k} \in \mathcal{M}_1(F_1(c_1), \dots, F_k(c_k), G(\otimes_{i=1}^k c_i))$ and thus $\mathcal{F}(\phi_{c_1, \dots, c_k}) \in \mathcal{M}_2(\mathcal{F}(F_1)(c_1), \dots, \mathcal{F}(F_k)(c_k), \mathcal{F}(G)(\otimes_{i=1}^k c_i))$. We define

$$\mathcal{F}_*(\phi) \in \mathcal{M}_2^{\mathcal{E}}(\mathcal{F}_*(F_1), \dots, \mathcal{F}_*(F_k); \mathcal{F}_*(G))$$

by

$$\mathcal{F}_*(\phi)_{c_1, \dots, c_k} = \mathcal{F}(\phi_{c_1, \dots, c_k}) \in \mathcal{M}_2(\mathcal{F}_*(F_1)(c_1), \dots, \mathcal{F}_*(F_k)(c_k); \mathcal{F}_*(G)(\otimes_{i=1}^k c_i)).$$

We need to show that $\mathcal{F}_*(\phi)$ satisfies the naturality condition; that is, if we are given morphisms $f_i : c_i \rightarrow c'_i$ in \mathcal{E} , we need to show that

$$\Gamma(\mathcal{F}_*(G)(\otimes f_i); \mathcal{F}_*(\phi)_{c_1, \dots, c_k}) = \Gamma(\mathcal{F}_*(\phi)_{c'_1, \dots, c'_k}; \mathcal{F}_*(F_1)(f_1), \dots, \mathcal{F}_*(F_k)(f_k)).$$

But we know that this is true for ϕ , thus we know that

$$\Gamma(G(\otimes f_i); \phi_{c_1, \dots, c_k}) = \Gamma(\phi_{c'_1, \dots, c'_k}; F_1(f_1), \dots, F_k(f_k)).$$

Applying \mathcal{F} to both sides and using that \mathcal{F} is a multifunctor we see that

$$\begin{aligned} \Gamma(\mathcal{F}_*(G)(\otimes f_i); \mathcal{F}_*(\phi)_{c_1, \dots, c_k}) &= \mathcal{F}(\Gamma(G(\otimes f_i); \phi_{c_1, \dots, c_k})) \\ \mathcal{F}(\Gamma(\phi_{c'_1, \dots, c'_k}; F_1(f_1), \dots, F_k(f_k))) &= \Gamma(\mathcal{F}_*(\phi)_{c'_1, \dots, c'_k}; \mathcal{F}_*(F_1)(f_1), \dots, \mathcal{F}_*(F_k)(f_k)), \end{aligned}$$

which proves that $\mathcal{F}_*(\phi)$ also satisfies the naturality condition.

Let's show now that \mathcal{F}_* respects the multiproduct. To do so consider F_{ir}, G_i and H objects in $\mathcal{M}_1^\mathcal{E}$ for $1 \leq i \leq k$ and $1 \leq r \leq j_i$. Also consider $\phi^i \in \mathcal{M}_1^\mathcal{E}(F_{i1}, \dots, F_{ij_i}, G_i)$ and $\psi \in \mathcal{M}_1^\mathcal{E}(G_1, \dots, G_k, H)$ for $1 \leq i \leq k$ and c_{ir} objects in \mathcal{E} . Then by definition

$$\begin{aligned} (\mathcal{F}_*\Gamma(\psi; \phi^1, \dots, \phi^k))_{c_{11}, \dots, c_{kj_k}} &= \mathcal{F}_*\Gamma(\psi_{\otimes_{r=1}^{j_1} c_{1r}, \dots, \otimes_{r=1}^{j_k} c_{kr}}; \phi_{c_{11}, \dots, c_{1j_1}}^1, \dots, \phi_{c_{k1}, \dots, c_{kj_k}}^k) \\ &= \Gamma(\mathcal{F}_*(\psi)_{\otimes_{r=1}^{j_1} c_{1r}, \dots, \otimes_{r=1}^{j_k} c_{kr}}; \mathcal{F}_*(\phi^1)_{c_{11}, \dots, c_{1j_1}}, \dots, \mathcal{F}_*(\phi^k)_{c_{k1}, \dots, c_{kj_k}}) \\ &= (\Gamma(\mathcal{F}_*(\psi); \mathcal{F}_*(\phi^1), \dots, \mathcal{F}_*(\phi^k)))_{c_{11}, \dots, c_{kj_k}}, \end{aligned}$$

and thus \mathcal{F}_* preserves the multiproduct. On the other hand, it's trivial to show that \mathcal{F}_* preserves units. We still need to show that \mathcal{F}_* is compatible with the action of Σ_k . Thus take $\sigma \in \Sigma_k$. We need to show that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{M}_1^\mathcal{E}(F_1, \dots, F_k; G) & \xrightarrow{\mathcal{F}_*} & \mathcal{M}_2^\mathcal{E}(\mathcal{F}_*(F_1), \dots, \mathcal{F}_*(F_k); \mathcal{F}_*(G)) \\ \sigma^* \downarrow & & \downarrow \sigma^* \\ \mathcal{M}_1^\mathcal{E}(F_{\sigma(1)}, \dots, F_{\sigma(k)}; G) & \xrightarrow{\mathcal{F}_*} & \mathcal{M}_2^\mathcal{E}(\mathcal{F}_*(F_{\sigma(1)}), \dots, \mathcal{F}_*(F_{\sigma(k)}); \mathcal{F}_*(G)). \end{array}$$

But by definition and the fact that \mathcal{F} is a multifunctor we have that

$$\begin{aligned} \mathcal{F}_*(\sigma^*\phi)_{c_{\sigma(1)}, \dots, c_{\sigma(k)}} &= \mathcal{F}(\sigma^*\phi_{c_{\sigma(1)}, \dots, c_{\sigma(k)}}) = \mathcal{F}(\sigma^*\Gamma(G(\tau_{\sigma, c_1, \dots, c_k}); \phi_{c_1, \dots, c_k})) \\ &= \sigma^*(\mathcal{F}_*(\Gamma(G(\tau_{\sigma, c_1, \dots, c_k}); \phi_{c_1, \dots, c_k}))) = \sigma^*(\Gamma(\mathcal{F}_*(G)(\tau_{\sigma, c_1, \dots, c_k}); \mathcal{F}_*(\phi_{c_1, \dots, c_k}))) \\ &= \sigma^*(\mathcal{F}_*(\phi))_{c_{\sigma(1)}, \dots, c_{\sigma(k)}}. \end{aligned}$$

This proves that \mathcal{F}_* is a multifunctor. In the case that $\mathcal{M}_1, \mathcal{M}_2$ are enriched over \mathbf{Cat} and \mathcal{F} and enriched multifunctor, then by a similar argument we see that \mathcal{F}_* is a multifunctor. \square

In (4) Elmendorff and Mandell constructed a multifunctor $K : \mathbb{P} \rightarrow \Sigma\mathcal{S}$. Using this multifunctor and the previous theorem we get the following corollary.

Corollary 23 *The multifunctor $K : \mathbb{P} \rightarrow \Sigma\mathcal{S}$ induces an enriched multifunctor $K_* : \mathbb{P}^\mathcal{E} \rightarrow \Sigma\mathcal{S}^\mathcal{E}$*

5 Multifunctors to $\mathbb{P}^{\mathcal{E}}$

In this section we study the multicategory $\mathbb{P}^{\mathcal{E}}$; that is, the multicategory obtained by taking $\mathcal{M} = \mathbb{P}$ in the previous construction. In particular we characterize the multifunctors to $\mathbb{P}^{\mathcal{E}}$ out of certain parameters multicategories Σ_* and $E\Sigma_*$ that are defined in this section. In (4) Elmendorf and Mandell proved that multifunctors $\Sigma_*, E\Sigma_* \rightarrow \mathbb{P}$ are determined by additional structure on the image of the only object of Σ_* and $E\Sigma_*$ respectively. Following their idea, we show that multifunctors from Σ_* and $E\Sigma_*$ to $\mathbb{P}^{\mathcal{E}}$ are determined by additional structure on the fibered category associated to the functor $\mathcal{E} \rightarrow \mathbb{P}$ that is the image of the only object of Σ_* and $E\Sigma_*$.

We will denote by Σ_* the associative operad; that is, Σ_* is the operad whose value at $k \geq 0$ is the symmetric group Σ_k . This operad is relevant as its algebras on a symmetric monoidal category are the associative monoids. We regard Σ_* as a multicategory with only one object. The goal of this section is to give an explicit description to the multifunctors

$$S : \Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}.$$

Such a multifunctor maps the unique object of Σ_* to an object F of $\mathbb{P}^{\mathcal{E}}$. The multifunctor $\Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}$, then factors through the multicategory generated by F in $\mathbb{P}^{\mathcal{E}}$. Suppose we have such a multifunctor $\Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}$ mapping the only object of Σ_* to F . Consider $1_2 \in \Sigma_2$ the identity element. This is a 2-morphism in the multicategory Σ_* , hence it is mapped to a 2-morphism $\otimes \in \mathbb{P}^{\mathcal{E}}(F, F; F)$. This operation is strictly associative, there is a unit 1 that is an object of $F(1_{\mathcal{E}})$. The operation \otimes and the unit 1 satisfy similar coherences as in (4, Definition 3.3). To be more precise, we have that $\otimes \in \mathbb{P}^{\mathcal{E}}(F, F; F)$, this means that for every pair $c_1, c_2 \in \mathcal{O}_{\mathcal{E}}$ we have a 2-linear map

$$\otimes_{c_1, c_2} : F(c_1) \times F(c_2) \rightarrow F(c_1 \otimes c_2)$$

Each \otimes_{c_1, c_2} comes equipped with distributivity maps

$$\begin{aligned} d_{c_1, c_2}^l &: (x \otimes_{c_1, c_2} y) \oplus (x' \otimes_{c_1, c_2} y) \rightarrow (x \oplus x') \otimes_{c_1, c_2} y, \\ d_{c_1, c_2}^r &: (x \otimes_{c_1, c_2} y) \oplus (x \otimes_{c_1, c_2} y') \rightarrow x \otimes_{c_1, c_2} (y \oplus y') \end{aligned}$$

that are the identity whenever x or x' equal 0_{c_1} or whenever y or y' equal 0_{c_2} .

By the definition of a 2-linear map the following coherences are satisfied.

$$\begin{array}{ccc}
(x \otimes_{c_1, c_2} y) \oplus (x' \otimes_{c_1, c_2} y) \oplus (x'' \otimes_{c_1, c_2} y) & \xrightarrow{d_{c_1, c_2}^l \oplus \text{id}} & ((x \oplus x') \otimes_{c_1, c_2} y) \oplus (x'' \otimes_{c_1, c_2} y) & \text{(c.1)} \\
\text{id} \oplus d_{c_1, c_2}^l \downarrow & & \downarrow d_{c_1, c_2}^l & \\
(x \otimes_{c_1, c_2} y) \oplus ((x' \oplus x'') \otimes_{c_1, c_2} y) & \xrightarrow{d_{c_1, c_2}^l} & (x \oplus x' \oplus x'') \otimes_{c_1, c_2} y &
\end{array}$$

is a commutative diagram and there is a similar commutative diagram for d_{c_1, c_2}^r .

Also

$$\begin{array}{ccc}
(x \otimes_{c_1, c_2} y) \oplus (x' \otimes_{c_1, c_2} y) & \xrightarrow{d_{c_1, c_2}^l} & (x \oplus x') \otimes_{c_1, c_2} y & \text{(c.2)} \\
\gamma^\oplus \downarrow & & \downarrow \gamma^\oplus \otimes_{c_1, c_2} \text{id} & \\
(x' \otimes_{c_1, c_2} y) \oplus (x \otimes_{c_1, c_2} y) & \xrightarrow{d_{c_1, c_2}^l} & (x' \oplus x) \otimes_{c_1, c_2} y &
\end{array}$$

is a commutative diagram and there is a similar commutative diagram for d_{c_1, c_2}^r .

We also need the following diagram to commute

$$\begin{array}{ccc}
& & (x \otimes_{c_1, c_2} (y \oplus y')) \oplus (x' \otimes_{c_1, c_2} (y \oplus y')) & \text{(c.3)} \\
& & \nearrow d_{c_1, c_2}^r \oplus d_{c_1, c_2}^r & \\
(x \otimes_{c_1, c_2} y) \oplus (x \otimes_{c_1, c_2} y') \oplus (x' \otimes_{c_1, c_2} y) \oplus (x' \otimes_{c_1, c_2} y') & & & \\
\downarrow \text{id} \oplus \gamma^\oplus \oplus \text{id} & & \searrow d_{c_1, c_2}^l & \\
(x \otimes_{c_1, c_2} y) \oplus (x' \otimes_{c_1, c_2} y) \oplus (x \otimes_{c_1, c_2} y') \oplus (x' \otimes_{c_1, c_2} y') & & (x \oplus x') \otimes_{c_1, c_2} (y \oplus y') & \\
& & \nearrow d_{c_1, c_2}^r & \\
& & ((x \oplus x') \otimes_{c_1, c_2} y) \oplus ((x \oplus x') \otimes_{c_1, c_2} y') & \\
& & \nwarrow d_{c_1, c_2}^l \oplus d_{c_1, c_2}^l &
\end{array}$$

In addition, we require that \otimes vanishes whenever one of the inputs is zero, more concretely, we require that

$$\begin{aligned}
0_{c_1} \otimes_{c_1, c_2} x &= 0_{c_1 \otimes c_2}, & \text{(c.4)} \\
x \otimes_{c_1, c_2} 0_{c_2} &= 0_{c_1 \otimes c_2}.
\end{aligned}$$

Diagrams (c.1), (c.2), (c.3) and (c.4) say that each \otimes_{c_1, c_2} is a 2-linear map. In

order for \otimes to be a 2-morphism in the multicategory $\mathbb{P}^{\mathcal{E}}$ we need the naturality condition to be satisfied; that is, given morphisms $f_i : c_i \rightarrow c'_i$ in \mathcal{E} for $i = 1$ and $i = 2$, then the following diagram is commutative

$$\begin{array}{ccc} F(c_1) \times F(c_2) & \xrightarrow{F(f_1) \times F(f_2)} & F(c'_1) \times F(c'_2) \\ \otimes_{c_1, c_2} \downarrow & & \downarrow \otimes_{c'_1, c'_2} \\ F(c_1 \otimes c_2) & \xrightarrow{F(f_1 \otimes f_2)} & F(c'_1 \otimes c'_2). \end{array} \quad (\mathbf{c.5})$$

We need this diagram to be a commutative diagram of 2-linear maps; that is, given x and x' objects in $F(c_1)$, y object in $F(c_2)$ we need the maps (*) and (**) to be the same

$$F(f_1)x \otimes_{c'_1, c'_2} F(f_2)y \oplus F(f_1)x' \otimes_{c'_1, c'_2} F(f_2)y \xrightarrow{d_{c'_1, c'_2}^l} F(f_1)x \oplus F(f_1)x' \otimes_{c'_1, c'_2} F(f_2)y \quad (*)$$

$$\xrightarrow{\lambda_{F(f_1)}} (F(f_1)(x \oplus x')) \otimes_{c'_1, c'_2} F(f_2)y,$$

$$F(f_1 \otimes f_2)(x \otimes_{c_1, c_2} y) \oplus F(f_1 \otimes f_2)(x' \otimes_{c_1, c_2} y) \xrightarrow{\lambda_{F(f_1 \otimes f_1)}} \quad (**)$$

$$F(f_1 \otimes f_2)(x \otimes_{c_1, c_2} y \oplus x' \otimes_{c_1, c_2} y) \xrightarrow{F(f_1 \otimes f_2)(d_{c_1, c_2}^l)} F(f_1 \otimes f_2)((x \oplus x') \otimes_{c_1, c_2} y);$$

that is, we have that

$$\lambda_{F(f_1)} \circ d_{c'_1, c'_2}^l = F(f_1 \otimes f_2)(d_{c_1, c_2}^l) \circ \lambda_{F(f_1 \otimes f_2)}, \quad (\mathbf{c.6})$$

and a similar condition is satisfied for d_{c_1, c_2}^r .

Also we have that 1, which is an object of $F(1_{\mathcal{E}})$, is a strict unit, in this case this means that

$$1 \otimes_{1, c} x = x = x \otimes_{c, 1} 1. \quad (\mathbf{c.7})$$

We also need strict associativity, which in this case this means that for every objects c_1, c_2 and c_3 of \mathcal{E} , and every objects x, y and z of $F(c_1), F(c_2)$ and $F(c_3)$ respectively, we have

$$(x \otimes_{c_1, c_2} y) \otimes_{c_1 \otimes c_2, c_3} z = x \otimes_{c_1, c_2 \otimes c_3} (y \otimes_{c_2, c_3} z). \quad (\mathbf{c.8})$$

In addition we need the following diagram (**c.9**) to commute and a similar for

d^r

$$\begin{array}{ccc}
& (x \otimes_{c_1, c_2 \otimes c_3} (y \otimes_{c_2, c_3} z)) \oplus (x' \otimes_{c_1, c_2 \otimes c_3} (y \otimes_{c_2, c_3} z)) & \text{(c.9)} \\
& \swarrow = & \searrow d^l_{c_1, c_2 \otimes c_3} \\
((x \otimes_{c_1, c_2} y) \otimes_{c_1 \otimes c_2, c_3} z) \oplus ((x' \otimes_{c_1, c_2} y) \otimes_{c_1 \otimes c_2, c_3} z) & & (x \oplus x') \otimes_{c_1, c_2 \otimes c_3} (y \otimes_{c_2, c_3} z). \\
\downarrow d^l_{c_1 \otimes c_2, c_3} & & \uparrow = \\
((x \otimes_{c_1, c_2} y) \oplus (x' \otimes_{c_1, c_2} y)) \otimes_{c_1 \otimes c_2, c_3} z & & \\
\downarrow d^l_{c_1, c_2 \otimes c_1, c_2 \otimes c_3} \text{id} & & \\
((x' \oplus x) \otimes_{c_1, c_2} y) \otimes_{c_1 \otimes c_2, c_3} z & &
\end{array}$$

Finally we need the diagram (c.10) to be commutative.

$$\begin{array}{ccc}
((x \otimes_{c_1, c_2} y) \otimes_{c_1 \otimes c_2, c_3} z) \oplus ((x \otimes_{c_1, c_2} y') \otimes_{c_1 \otimes c_2, c_3} z) & \xrightarrow{d^l_{c_1 \otimes c_2, c_3}} & ((x \otimes_{c_1, c_2} y) \oplus (x \otimes_{c_1, c_2} y')) \otimes_{c_1 \otimes c_2, c_3} z \\
\downarrow = & & \downarrow d^r_{c_1, c_2 \otimes c_1 \otimes c_2, c_3} \text{id} \\
(x \otimes_{c_1, c_2 \otimes c_3} (y \otimes_{c_2, c_3} z)) \oplus (x \otimes_{c_1, c_2 \otimes c_3} (y' \otimes_{c_2, c_3} z)) & & (x \otimes_{c_1, c_2} (y \oplus y')) \otimes_{c_1 \otimes c_2, c_3} z \\
\downarrow d^r_{c_1, c_2 \otimes c_3} & & \downarrow = \\
x \otimes_{c_1, c_2 \otimes c_3} ((y \otimes_{c_2, c_3} z) \oplus (y' \otimes_{c_2, c_3} z)) & \xrightarrow{\text{id}_{c_1, c_2 \otimes c_3} d^l_{c_2, c_3}} & x \otimes_{c_1, c_2 \otimes c_3} ((y \oplus y') \otimes_{c_2, c_3} z). \\
& & \text{(c.10)}
\end{array}$$

We have the following theorem.

Theorem 24 *Having a multifunctor $\Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}$ mapping the only object of Σ_* to a functor $F : \mathcal{E} \rightarrow \mathbb{P}$ is equivalent to having a functor $F : \mathcal{E} \rightarrow \mathbb{P}$, a functor*

$$\otimes_{c_1, c_2} : F(c_1) \times F(c_2) \rightarrow F(c_1 \otimes c_2)$$

for each pair $c_1, c_2 \in \mathcal{O}_{\mathcal{E}}$, a unit 1 which is an object in $F(1_{\mathcal{E}})$ and distributivity maps

$$\begin{aligned}
d^l_{c_1, c_2} &: (x \otimes_{c_1, c_2} y) \oplus (x' \otimes_{c_1, c_2} y) \rightarrow (x \oplus x') \otimes_{c_1, c_2} y, \\
d^r_{c_1, c_2} &: (x \otimes_{c_1, c_2} y) \oplus (x \otimes_{c_1, c_2} y') \rightarrow x \otimes_{c_1, c_2} (y \oplus y')
\end{aligned}$$

satisfying conditions (c.1) – (c.10).

Proof: First suppose that we have a multifunctor $S : \Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}$. Then S

factors through the multicategory generated by $F : \mathcal{E} \rightarrow \mathbb{P}$, for some F . Thus, we may regard S as a multifunctor $S : \Sigma_* \rightarrow \{\mathbb{P}^{\mathcal{E}}(F, \dots, F; F)\}$, sending the only object of Σ_* to F . Let $1_n \in \Sigma_n$ be the identity element. Then 1_2 is a 2-morphism and we let $\otimes = S_2(1_2) \in \mathbb{P}^{\mathcal{E}}(F, F; F)$. Unraveling the definitions, we see that \otimes_{c_1, c_2} is 2-linear map for every $c_1, c_2 \in \mathcal{O}_{\mathcal{E}}$. Therefore conditions **(c.1)** – **(c.4)** hold. In addition, properties **(c.5)** and **(c.6)** hold as \otimes satisfies the naturality condition, this follows from the fact that \otimes is a 2-morphism in the multicategory $\mathbb{P}^{\mathcal{E}}$. Consider $1_0 \in \Sigma_0$. This element is mapped under S_0 to a 0-morphism in $\mathbb{P}^{\mathcal{E}}(; F)$. By definition, this is just an object in the category $F(1_{\mathcal{E}})$ which we define as the unit 1. Condition **(c.7)** follows from the fact that, in Σ_* , we have the equality $\Gamma_{\Sigma_*}(1_2; 1_1, 1_0) = 1_1 = \Gamma_{\Sigma_*}(1_2; 1_0, 1_1)$. Property **(c.8)**, the associativity property, follows from the fact that in the multicategory Σ_* , we have $\Gamma_{\Sigma_*}(1_2; 1_2, 1_1) = \Gamma_{\Sigma_*}(1_2; 1_1, 1_2)$, thus we have that

$$\Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; \otimes, 1_F) = \Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; 1_F, \otimes).$$

The previous equation implies condition **(c.8)**. Note that both $\Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; \otimes, 1_F)$ and $\Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; 1_F, \otimes)$ are 3-morphisms in $\mathbb{P}^{\mathcal{E}}$. Therefore, for every triple $c_1, c_2, c_3 \in \mathcal{O}_{\mathcal{E}}$,

$$\Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; \otimes, 1_F)_{c_1, c_2, c_3} \quad \text{and} \quad \Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; 1_F, \otimes)_{c_1, c_2, c_3},$$

are 3-linear maps that agree, in particular their structural maps δ^i must agree. This shows that conditions **(c.9)** and **(c.10)** are satisfied.

On the other hand, suppose we have a functor $F : \mathcal{E} \rightarrow \mathbb{P}$, a collection of 2-linear maps \otimes_{c_1, c_2} for each pair c_1 and $c_2 \in \mathcal{O}_{\mathcal{E}}$ and a unit 1 that satisfy properties **(c.1)** – **(c.10)**. We want to define a multifunctor $S : \Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}$. To begin, we define S evaluated on the only object of Σ_* as F .

Note that conditions **(c.1)** – **(c.4)** say precisely that \otimes_{c_1, c_2} is a 2-linear map

$$\otimes_{c_1, c_2} : F(c_1) \times F(c_2) \rightarrow F(c_1 \otimes c_2),$$

with structural maps d_{c_1, c_2}^l and d_{c_1, c_2}^r for every pair c_1 and $c_2 \in \mathcal{O}_{\mathcal{E}}$. In addition, by properties **(c.5)** and **(c.6)** we see that \otimes satisfies the naturality condition, therefore \otimes is a 2-morphism in $\mathbb{P}^{\mathcal{E}}(F, F; F)$. We claim that properties **(c.7)** – **(c.8)** imply that, as 3-morphisms in $\mathbb{P}^{\mathcal{E}}(F, F, F; F)$,

$$\Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; \otimes, 1_F) = \Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; 1_F, \otimes). \quad (5)$$

Indeed, for c_1, c_2 and $c_3 \in \mathcal{O}_{\mathcal{E}}$ we have

$$\begin{aligned} \Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; \otimes, 1_F)_{c_1, c_2, c_3} &= \otimes_{c_1 \otimes c_2, c_3} \circ (\otimes_{c_1, c_2} \times \text{id}_{F(c_3)}), \\ \Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; 1_F, \otimes)_{c_1, c_2, c_3} &= \otimes_{c_1, c_2 \otimes c_3} \circ (\text{id}_{F(c_1)} \times \otimes_{c_2, c_3}). \end{aligned}$$

By equation **(c.8)**, $\Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; \otimes, 1_F)_{c_1, c_2, c_3}$ and $\Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; 1_F, \otimes)_{c_1, c_2, c_3}$ agree as functors. On the other hand, the 3-linear map $\Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; \otimes, 1_F)_{c_1, c_2, c_3}$ has structural

maps $\delta^1, \delta^2, \delta^3$, where δ^1 is given by

$$\delta^1 = (d_{c_1, c_2}^l \otimes_{c_1 \otimes c_2, c_3} \text{id}) \circ d_{c_1 \otimes c_2, c_3}^l. \quad (6)$$

Similarly we can find equations δ^2 and δ^3 . On the other hand, the 3-linear map $\Gamma_{\mathbb{P}^{\mathcal{E}}}(\otimes; 1_F, \otimes)_{c_1, c_2, c_3}$ has structural maps $\delta_*^1, \delta_*^2, \delta_*^3$, where δ_*^1 is given by

$$\delta_*^1 = d_{c_1, c_2 \otimes c_3}^l. \quad (7)$$

Similarly we can find equations δ_*^2 and δ_*^3 . By diagram (c.9), we see that (6) and (7) agree. The same statement is true for δ_2, δ_3 and δ_*^2, δ_*^3 and it follows by conditions (c.9) and (c.10).

To construct the multifunctor S , we need to define for $k \geq 0$ a functor

$$S_k : \Sigma_k \rightarrow \mathbb{P}^{\mathcal{E}}(F, \dots, F; F)$$

that respects the action of Σ_k and the multiproduct. As Σ_k is a discrete category, it suffices to define $S_k(\sigma)$ for all $\sigma \in \Sigma_k$. We will define first $S_k(1_k)$ for $k \geq 0$. For $k = 0$ we define $S_0(1_0) = 1$. To define $S_k(1_k)$ for $k \geq 1$, we will use the following notation. Suppose that $c_1, \dots, c_k \in \mathcal{O}_{\mathcal{E}}$ and that x_i is an object in the permutative category $F(c_i)$, for $1 \leq i \leq k$. Define

$$x_{1, \dots, k} = x_1 \otimes \cdots \otimes x_k = (\cdots ((x_1 \otimes_{c_1, c_2} x_2) \otimes_{c_1 \otimes c_2, c_3} x_3) \cdots \otimes_{c_1 \otimes \cdots \otimes c_{k-1}, c_k} x_k).$$

Note that by condition (c.8), this is well defined and equals any rearranging of the parenthesis using the respective operations \otimes . With this in mind, define

$$S_k(1_k)_{c_1, \dots, c_k}(x_1, \dots, x_k) = x_1 \otimes \cdots \otimes x_k = x_{1, \dots, k}.$$

We need to define the structural maps of the $S_k(1_k)_{c_1, \dots, c_k}$'s in order for it to be a k -linear map. To do this, note that by condition (c.8) we have that

$$S_k(1_k)_{c_1, \dots, c_k}(x_1, \dots, x_k) = x_{1, \dots, k} = x_{1, \dots, i-1} \otimes x_i \otimes x_{i+1, \dots, k}.$$

Then we define

$$\begin{aligned} \delta^i : S_k(1_k)_{c_1, \dots, c_k}(x_1, \dots, x_i, \dots, x_k) \oplus S_k(1_k)_{c_1, \dots, c_k}(x_1, \dots, x'_i, \dots, x_k) \\ = x_{1, \dots, i-1} \otimes x_i \otimes x_{i+1, \dots, k} \oplus x_{1, \dots, i-1} \otimes x'_i \otimes x_{i+1, \dots, k} \\ \rightarrow x_{1, \dots, i-1} \otimes (x_i \oplus x'_i) \otimes x_{i+1, \dots, k} \\ = S_k(1_k)_{c_1, \dots, c_k}(x_1, \dots, x_i \oplus x'_i, \dots, x_k), \end{aligned}$$

to be the diagonal map in the following commutative diagram

$$\begin{array}{ccc}
x_{1,\dots,i-1} \otimes x_i \otimes x_{i+1,\dots,k} \oplus x_{1,\dots,i-1} \otimes x'_i \otimes x_{i+1,\dots,k} & & \\
\downarrow d^r & \searrow d^i & \\
& (x_{1,\dots,i-1} \otimes x_i \oplus x_{1,\dots,i-1} \otimes x'_i) \otimes x_{i+1,\dots,k} & \\
& \downarrow d^r \otimes \text{id} & \\
x_{1,\dots,i-1} \otimes (x_i \otimes x_{i+1,\dots,k} \oplus x'_i \otimes x_{i+1,\dots,k}) & & \\
& \searrow \text{id} \otimes d^i & \\
& x_{1,\dots,i-1} \otimes (x_i \oplus x'_i) \otimes x_{i+1,\dots,k} &
\end{array}$$

The previous diagram is commutative by condition **(c.10)** and by a direct computation we see that the condition of commutation with \oplus for δ^i is satisfied.

In general, we define for $\sigma \in \Sigma_k$

$$S_k(\sigma) = \sigma^* S_k(1_k).$$

This way defined, S_k is clearly respects the Σ_k action. Let's see that it respect the multiproduct. To do so, take $\phi_i \in \Sigma_{j_i}$ for $1 \leq i \leq k$ and $\sigma \in \Sigma_k$. We need to verify the equality

$$S_{j_1+\dots+j_k}(\Gamma_{\Sigma_*}(\sigma; \phi_1, \dots, \phi_k)) = \Gamma_{\mathbb{P}\mathcal{E}}(S_k(\sigma); S_{j_1}(\phi_1), \dots, S_{j_k}(\phi_k)).$$

In Σ_* , the multiproduct is defined by

$$\Gamma_{\Sigma_*}(\sigma; \phi_1, \dots, \phi_k) = \sigma_{\langle j_1, \dots, j_k \rangle} \circ (\phi_1 \oplus \dots \oplus \phi_k).$$

Here, $\phi_1 \oplus \dots \oplus \phi_k$ means the image of ϕ_1, \dots, ϕ_k under the canonical embedding

$$\Sigma_{j_1} \times \dots \times \Sigma_{j_k} \rightarrow \Sigma_{j_1+\dots+j_k}.$$

By definition we have

$$\begin{aligned}
S_{j_1+\dots+j_k}(\Gamma_{\Sigma_*}(\sigma; \phi_1, \dots, \phi_k)) &= (\Gamma_{\Sigma_*}(\sigma; \phi_1, \dots, \phi_k))^* S_{j_1+\dots+j_k}(1_{j_1+\dots+j_k}) \\
&= (\sigma_{\langle j_1, \dots, j_k \rangle} \circ (\phi_1 \oplus \dots \oplus \phi_k))^* S_{j_1+\dots+j_k}(1_{j_1+\dots+j_k}).
\end{aligned}$$

On the other hand,

$$\Gamma_{\mathbb{P}\mathcal{E}}(S_k(\sigma); S_{j_1}(\phi_1), \dots, S_{j_k}(\phi_k)) = \Gamma_{\mathbb{P}\mathcal{E}}(\sigma^* S_k(1_k); \phi_1^* S_{j_1}(1_{j_1}), \dots, \phi_k^* S_{j_k}(1_{j_k})).$$

Using properties Multi(3) and Multi(4) in the multicategory $\mathbb{P}\mathcal{E}$, we obtain

$$\begin{aligned}
&\Gamma_{\mathbb{P}\mathcal{E}}(S_k(\sigma); S_{j_1}(\phi_1), \dots, S_{j_k}(\phi_k)) \\
&= (\sigma_{\langle j_1, \dots, j_k \rangle} \circ (\phi_1 \oplus \dots \oplus \phi_k))^* \Gamma_{\mathbb{P}\mathcal{E}}(S_k(1_k); S_{j_1}(1_{j_1}), \dots, S_{j_k}(1_{j_k})).
\end{aligned}$$

Thus it suffices to show that

$$S_{j_1+\dots+j_k}(1_{j_1+\dots+j_k}) = \Gamma_{\mathbb{P}^{\mathcal{E}}}(S_k(1_k); S_{j_1}(1_{j_1}), \dots, S_{j_k}(1_{j_k})).$$

But this follows by induction and equation (5). Since S respects identities by definition, we see that S is a multifunctor mapping the only object of Σ_* to F . Finally, the fact that these correspondences are inverses of each other follows by a direct computation. \square

We now discuss the case where the operations \otimes_{c_1, c_2} , satisfy some sort of commutativity up to coherent isomorphisms. Let's denote by $E\Sigma_*$ the category valued operad that for each j , corresponds $E\Sigma_j$ the translation category of Σ_j ; that is, the object set of $E\Sigma_j$ is Σ_j and there is exactly one morphism between any pair of objects of $E\Sigma_j$. We can see $E\Sigma_*$ as a multicategory with only one object and that is enriched over \mathbf{Cat} . In the same spirit as in the previous theorem, we want to give an explicit description to the enriched multifunctors

$$S : E\Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}.$$

Such a multifunctor maps the unique object of $E\Sigma_*$ to an object F of $\mathbb{P}^{\mathcal{E}}$. The multifunctor $E\Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}$, then factors through the multicategory generated by F in $\mathbb{P}^{\mathcal{E}}$. We can see each Σ_j as the trivial category with only the identity morphisms. Thus we have an inclusion functor $i_j : \Sigma_j \rightarrow E\Sigma_j$ for every $j \geq 0$. The collection of all i_j gives rise to a multifunctor

$$i : \Sigma_* \rightarrow E\Sigma_*.$$

This multifunctor is trivially enriched over \mathbf{Cat} . By composing S with i , we see that for F conditions **(c.1)** – **(c.10)** are satisfied. In addition, we also have the following properties. Let $\xi \in \Sigma_2$ is the map that permutes the two elements of $\{1, 2\}$. Then we can find a map of 2-morphisms

$$\mu : \otimes \rightarrow \xi^* \otimes;$$

that is, for each pair of objects c_1 and c_2 of \mathcal{E} , we have a map of 2-linear maps

$$\mu_{c_1, c_2} : x \otimes_{c_1, c_2} y \rightarrow F(\tau_{\xi, c_2, c_1})(y \otimes_{c_2, c_1} x)$$

satisfying the following coherence diagrams

$$\begin{array}{ccc} x \otimes_{c_1, c_2} 0_{c_2} & \xrightarrow{\mu_{c_1, c_2}} & F(\tau_{\xi, c_2, c_1})(0_{c_2} \otimes_{c_2, c_1} x) \\ \downarrow = & & \downarrow = \\ 0_{c_1 \otimes c_2} & \xrightarrow{\equiv} & F(\tau_{\xi, c_2, c_1})(0_{c_2 \otimes c_1}), \end{array} \quad (\mathbf{c.11})$$

$$\begin{array}{ccc}
x \otimes_{c_1, c_2} y & \xrightarrow{\mu_{c_1, c_2}} & F(\tau_{\xi, c_2, c_1})(y \otimes_{c_2, c_1} x) \\
& \searrow = & \downarrow F(\tau_{\xi, c_2, c_1})(\mu_{c_2, c_1}) \\
& & F(\tau_{\xi, c_2, c_1})F(\tau_{\xi, c_1, c_2})(x \otimes_{c_1, c_2} y),
\end{array} \tag{c.12}$$

$$\begin{array}{ccc}
& & (x \oplus x') \otimes_{c_1, c_2} y \\
& \nearrow d_{c_1, c_2}^l & \downarrow \mu_{c_1, c_2} \\
(x \otimes_{c_1, c_2} y) \oplus (x' \otimes_{c_1, c_2} y) & & F(\tau_{\xi, c_2, c_1})(y \otimes_{c_2, c_1} (x \oplus x')), \\
\downarrow \mu_{c_1, c_2} \oplus \mu_{c_1, c_2} & & \uparrow F(\tau_{\xi, c_2, c_1})(d_{c_2, c_1}^r) \\
F(\tau_{\xi, c_2, c_1})(y \otimes_{c_2, c_1} x) \oplus F(\tau_{\xi, c_2, c_1})(y \otimes_{c_2, c_1} x') & & \\
& \searrow \lambda_{F(\tau_{\xi, c_2, c_1})} & \\
& & F(\tau_{\xi, c_2, c_1})((y \otimes_{c_2, c_1} x) \oplus (y \otimes_{c_2, c_1} x'))
\end{array} \tag{c.13}$$

$$\begin{array}{ccc}
x \otimes_{c_1, c_2 \otimes c_3} (y \otimes_{c_2, c_3} z) & \xrightarrow{\text{id}_{c_1, c_2 \otimes c_3} \otimes \mu_{c_2, c_3}} & x \otimes_{c_1, c_2 \otimes c_3} (F(\tau_{\xi, c_3, c_2})z \otimes_{c_3, c_2} y) \\
\downarrow = & & \downarrow = \\
(x \otimes_{c_1, c_2} y) \otimes_{c_1 \otimes c_2, c_3} z & & F(1 \otimes \tau_{\xi, c_3, c_2})(x \otimes_{c_1, c_3 \otimes c_2} (z \otimes_{c_3, c_2} y)) \\
\downarrow \mu_{c_1 \otimes c_2, c_3} & & \downarrow = \\
F(\tau_{\xi, c_3, c_1 \otimes c_2})(z \otimes_{c_3, c_1 \otimes c_2} (x \otimes_{c_1, c_2} y)) & & F(1 \otimes \tau_{\xi, c_3, c_2})((x \otimes_{c_1, c_3} z) \otimes_{c_1 \otimes c_3, c_2} y) \\
\downarrow = & & \downarrow F(\text{id} \otimes \tau_{\xi, c_3, c_2})(\mu_{c_1, c_3}) \otimes \text{id} \\
F(\tau_{\xi, c_3, c_1 \otimes c_2})((z \otimes_{c_3, c_1} x) \otimes_{c_3 \otimes c_1, c_2} y) & \xrightarrow{=} & F(1 \otimes \tau_{\xi, c_3, c_2})(F(\tau_{\xi, c_3, c_1})(z \otimes_{c_3, c_1} x) \otimes_{c_1 \otimes c_3, c_2} y).
\end{array} \tag{c.14}$$

The equalities in the previous diagram are obtained by applying the naturality property of \otimes and the associativity property of \otimes which are satisfied as conditions (c.1 – c.10) hold. We have the following theorem.

Theorem 25 *Having a multifunctor $E\Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}$ mapping the only object of $E\Sigma_*$ to a functor $F : \mathcal{E} \rightarrow \mathbb{P}$ is equivalent to having a functor $F : \mathcal{E} \rightarrow \mathbb{P}$, for each pair c_1, c_2 a functor*

$$\otimes_{c_1, c_2} : F(c_1) \times F(c_2) \rightarrow F(c_1 \otimes c_2),$$

a unit 1 which is an object in $F(1_{\mathcal{E}})$, distributivity maps

$$\begin{aligned} d_{c_1, c_2}^l &: (x \otimes_{c_1, c_2} y) \oplus (x' \otimes_{c_1, c_2} y) \rightarrow (x \oplus x') \otimes_{c_1, c_2} y \\ d_{c_1, c_2}^r &: (x \otimes_{c_1, c_2} y) \oplus (x \otimes_{c_1, c_2} y') \rightarrow x \otimes_{c_1, c_2} (y \oplus y') \end{aligned}$$

and a natural transformation

$$\mu_{c_1, c_2} : x \otimes_{c_1, c_2} y \rightarrow F(\tau_{\xi, c_2, c_1})(y \otimes_{c_2, c_1} x)$$

satisfying conditions **(c.1)** – **(c.14)**.

Proof: Suppose that $S : E\Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}$ is an enriched multifunctor sending the only object of $E\Sigma_*$ to a functor $F : \mathcal{E} \rightarrow \mathbb{P}$. Let $\otimes = S_2(1_2)$, μ the image of the unique morphism in $E\Sigma_2$ between 1_2 and ξ under S_2 and 1 the image of 1_0 under S_0 . Then we see by the way we derived conditions **(c.1)** – **(c.14)** that these are satisfied by $F, \otimes, \xi, 1$.

On the other hand, suppose we have a functor $F : \mathcal{E} \rightarrow \mathbb{P}$, a collection of 2-linear maps \otimes_{c_1, c_2} , natural transformations μ_{c_1, c_2} for each pair c_1 and $c_2 \in \mathcal{O}_{\mathcal{E}}$ and a unit 1 that satisfy properties **(c.1)** – **(c.14)**. We want to define a multifunctor $S : E\Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}$. To begin, note that in particular conditions **(c.1)** – **(c.10)** are satisfied and thus by Theorem 24 we have a multifunctor $S : \Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}$ sending the only object of Σ_* to F . We want to see that this multifunctor extends to a multifunctor $S : E\Sigma_* \rightarrow \mathbb{P}^{\mathcal{E}}$. We claim that the natural transformations $\{\mu_{c_1, c_2}\}$, form a map of 2-morphisms from \otimes to $\xi^* \otimes$. Indeed, if c_1 and $c_2 \in \mathcal{O}_{\mathcal{E}}$, and x_i is an object in the category $F(c_i)$ for $i = 1, 2$, then μ_{c_1, c_2} is a natural map

$$\mu_{c_1, c_2}(x_1, x_2) : x_1 \otimes_{c_1, c_2} x_2 \rightarrow (\xi^* \otimes_{c_1, c_2})(x_1, x_2).$$

Also, $\mu_{c_1, c_2}(x_1, x_2)$ is the identity whenever x_1 or x_2 is zero. This is a consequence of condition **(c.11)**. In order to conclude that μ_{c_1, c_2} is a map of 2-linear maps, we need to show that the following diagram is commutative

$$\begin{array}{ccc} x_1 \otimes_{c_1, c_2} x_2 \oplus x'_1 \otimes_{c_1, c_2} x_2 & \xrightarrow{d_{c_1, c_2}^l} & (x_1 \oplus x'_1) \otimes_{c_1, c_2} x_2 \\ \mu_{c_1, c_2} \oplus \mu_{c_1, c_2} \downarrow & & \downarrow \mu_{c_1, c_2} \\ (\xi^* \otimes_{c_1, c_2})(x_1, x_2) \oplus (\xi^* \otimes_{c_1, c_2})(x'_1, x_2) & \xrightarrow{\delta_{\xi^* \otimes_{c_1, c_2}}^1} & (\xi^* \otimes_{c_1, c_2})(x_1 \oplus x'_1, x_2) \end{array}$$

and a similar one for $\delta_{\xi^* \otimes_{c_1, c_2}}^2$. But by definition,

$$\delta_{\xi^* \otimes_{c_1, c_2}}^1 = F(\tau_{\xi, c_2, c_1})(d_{c_2, c_1}^r) \circ \lambda_{F(\tau_{\xi, c_2, c_1})},$$

and thus the commutativity of the previous diagram follows from condition **(c.13)**. We need to define a functor $S_k : E\Sigma_k \rightarrow \mathbb{P}^{\mathcal{E}}(F, \dots, F; F)$. Suppose $\rho, \sigma \in \Sigma_k$ and $f : \rho \rightarrow \sigma$ is the only morphism in $E\Sigma_k$ from ρ to σ . Using the action of Σ_k we only need to define $S_k(f)$ in the case of $\rho = 1_k$. Thus we need to define a morphism on the category $\mathbb{P}^{\mathcal{E}}(F, \dots, F; F)$ from $S_k(1_k)$ to $S_k(\sigma)$. Let's consider first the case where σ is a transposition. Thus assume $\sigma = (m, n)$, where $1 \leq m < n \leq k$. Take $c_1, \dots, c_k \in \mathcal{O}_{\mathcal{E}}$ and x_i an object of $F(c_i)$ for $i = 1, \dots, k$. By definition we have

$$S_k(\sigma)_{c_1, \dots, c_k}(x_1, \dots, x_k) = F(\tau_{(m,n), c_i, c_j})(x_{1, \dots, m-1} \otimes x_m \otimes x_{m+1, \dots, n-1} \otimes x_n \otimes x_{n+1, \dots, k})$$

Define $S_k(f)_{c_1, \dots, c_k}$ to be the following composite in the category $\mathbb{P}^{\mathcal{E}}(F, \dots, F; F)$

$$\begin{aligned} x_{1, \dots, k} &= x_{1, \dots, m-1} \otimes x_m \otimes x_{m+1, \dots, n-1} \otimes x_n \otimes x_{n+1, \dots, k} \\ &\rightarrow F(\text{id} \otimes \tau_{\xi} \otimes \text{id})(x_{1, \dots, m-1} \otimes x_{m+1, \dots, n-1} \otimes x_m \otimes x_n \otimes x_{n+1, \dots, k}) \\ &\rightarrow F(\text{id} \otimes \tau_{\xi} \otimes \text{id})F(\text{id} \otimes \tau_{\xi} \otimes \text{id}) \\ &\quad (x_{1, \dots, m-1} \otimes x_{m+1, \dots, n-1} \otimes x_n \otimes x_m \otimes x_{n+1, \dots, k}) \\ &\rightarrow F(\text{id} \otimes \tau_{\xi} \otimes \text{id})F(\text{id} \otimes \tau_{\xi} \otimes \text{id})F(\text{id} \otimes \tau_{\xi} \otimes \text{id}) \\ &\quad (x_{1, \dots, m-1} \otimes x_n \otimes x_{m+1, \dots, n-1} \otimes x_m \otimes x_{n+1, \dots, k}) \\ &= F(\tau_{\sigma})(x_{\sigma(1), \dots, \sigma(k)}) \end{aligned}$$

By condition **(c.14)**, this is well defined and agrees with any other composition of maps obtained by applying μ to a sequence that starts in $x_{1, \dots, k}$ and finish in $F(\tau_{\sigma, c_1, \dots, c_k})(x_{\sigma(1), \dots, \sigma(k)})$. As any element of Σ_k can be obtained by composition of transpositions, we see that we can extend this definition by using the Σ_k action. This way we define $S_k(f)$, where f is a morphism in $E\Sigma_k$ from 1_k to σ . Using the Σ_k action we can extend the definition to all the morphisms in $E\Sigma_k$. We need to verify that this way defined, we obtain functors

$$S_k : E\Sigma_k \rightarrow \mathbb{P}^{\mathcal{E}}(F, \dots, F; F).$$

This is done by induction on k . For $k = 0$ and $k = 1$ it is trivial as $E\Sigma_i$ is a discrete category. For $k = 2$ there are only two morphisms that are not the identity, namely, $f : 1_2 \rightarrow \xi$ and its inverse. By condition **(c.12)** we see that $S_2(f) \circ S_2(f^{-1}) = S_2(f^{-1}) \circ S_2(f) = \text{id}$. The case where $k \geq 2$ follows easily by induction using condition **(c.14)** and is left for the reader. We are left to prove that S_k respects the multiproduct on the level of morphisms of the categories $E\Sigma_k$ and $\mathbb{P}^{\mathcal{E}}(F, \dots, F; F)$. To see this, note that by the Σ_k action and using properties Multi(3)-Multi(4) as in Theorem 24 we only need to prove that

$$\Gamma_{\mathbb{P}^{\mathcal{E}}}(S_k(g); S_{j_1}(f_1), \dots, S_{j_k}(f_r)) = S_{j_1 + \dots + j_k}(\Gamma_{E\Sigma_*}(g; f_1, \dots, f_k))$$

in the case where $f_i : 1_{j_i} \rightarrow \sigma_i$, and $g : 1_k \rightarrow \rho$. But for each $c_{ir} \in \mathcal{O}_{\mathcal{E}}$, for $1 \leq i \leq k$ and $1 \leq r \leq j_i$ we have that both

$$\Gamma_{\mathbb{P}^{\mathcal{E}}}(S_k(g); S_{j_1}(f_1), \dots, S_{j_k}(f_r))_{c_{11}, \dots, c_{kj_k}} \quad \text{and} \quad S_{j_1 + \dots + j_k}(\Gamma_{E\Sigma_*}(g; f_1, \dots, f_k))_{c_{11}, \dots, c_{kj_k}}$$

are obtained by successive applications of μ starting and ending in the same place. By (c.14) it follows that these morphism have to agree. Finally, the fact that these correspondences are inverses of each other follows by a direct computation. \square

The application of Theorem 25 that we are looking for is the following theorem.

Theorem 26 *If $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ is a fibered bipermutative category then for the associated functor $\Psi : \mathcal{A} \rightarrow \mathbb{P}$, we can find functors*

$$\otimes_{\underline{u}, \underline{v}} : \Psi(\underline{u}) \times \Psi(\underline{v}) \rightarrow \Psi(\underline{u} \odot \underline{v})$$

for each pair \underline{u} and \underline{v} of objects of \mathcal{A} , a unit $\mathbf{1}$ which is an object in $\Psi(\odot)$, distributivity maps

$$\begin{aligned} d_{\underline{u}, \underline{v}}^l &: (F \otimes_{\underline{u}, \underline{v}} G) \oplus (F' \otimes_{\underline{u}, \underline{v}} G) \rightarrow (F \oplus F') \otimes_{\underline{u}, \underline{v}} G, \\ d_{\underline{u}, \underline{v}}^r &: (F \otimes_{\underline{u}, \underline{v}} G) \oplus (F \otimes_{\underline{u}, \underline{v}} G') \rightarrow F \otimes_{\underline{u}, \underline{v}} (G \oplus G') \end{aligned}$$

and a natural transformation

$$\mu_{\underline{u}, \underline{v}} : F \otimes_{\underline{u}, \underline{v}} G \rightarrow \Psi(\tau_{\xi, \underline{u}, \underline{v}})(G \otimes_{\underline{v}, \underline{u}} F)$$

satisfying conditions (c.1) – (c.14) of Theorem 25.

Proof: Let's begin by defining $\otimes_{\underline{u}, \underline{v}}$. Take $\underline{u} = (u_1, \dots, u_n)$ and $\underline{v} = (v_1, \dots, v_m)$ and

$$F = \sum_{i=1}^r F_{i1} \otimes \dots \otimes F_{in}, \quad \text{and} \quad G = \sum_{j=1}^s G_{j1} \otimes \dots \otimes G_{jm},$$

objects of $\Psi(\underline{u})$ and $\Psi(\underline{v})$ respectively. Define

$$F \otimes_{\underline{u}, \underline{v}} G = \sum_{j=1}^s \sum_{i=1}^r F_{i1} \otimes \dots \otimes F_{in} \otimes G_{j1} \otimes \dots \otimes G_{jm}.$$

In the case where $\underline{v} = ()$, we define

$$F \otimes_{\underline{u}, ()} G = \sum_{j=1}^s \sum_{i=1}^r F_{i1} \otimes \dots \otimes (F_{in} \otimes G).$$

Note that every object of $\mathcal{C}/\underline{u} \odot \underline{v}$ can be written in the form $\underline{f} \odot \underline{g} = (f_1, \dots, f_n, g_1, \dots, g_m)$, where $\underline{f} = (f_1, \dots, f_n)$ is an object of $\mathcal{C}/\underline{u}$ and $\underline{g} = (g_1, \dots, g_m)$ and object of $\mathcal{C}/\underline{v}$. Then using the distributivity maps d^l and d^r of \mathcal{D} we can obtain a coherent isomorphism

$$\lambda_{\underline{f}, \underline{g}} : F(\underline{f}) \otimes G(\underline{g}) \xrightarrow{\cong} (F \otimes_{\underline{u}, \underline{v}} G)(\underline{f} \odot \underline{g}).$$

For morphisms $\alpha : F \rightarrow F'$ and $\beta : G \rightarrow G'$, define $\alpha \otimes \beta$ to be the base point preserving natural transformation defined by the following diagram

$$\begin{array}{ccc} (F \otimes_{\underline{u}, \underline{v}} G)(\underline{f} \odot \underline{g}) & \xrightarrow{(\alpha \otimes \beta)_{\underline{f} \odot \underline{g}}} & (F' \otimes G')(\underline{f} \odot \underline{g}) \\ \lambda_{\underline{f}, \underline{g}}^{-1} \downarrow & & \uparrow \lambda_{\underline{f}, \underline{g}} \\ F(\underline{f}) \otimes G(\underline{g}) & \xrightarrow{\alpha_{\underline{f}} \otimes \beta_{\underline{g}}} & F'(\underline{f}) \otimes G'(\underline{g}). \end{array}$$

Using the coherence of the distributivity map and the fact that these preserve the base point, we see that this way defined $\alpha \otimes \beta$ is a base point preserving natural transformation. This defines the functor $\otimes_{\underline{u}, \underline{v}}$.

We also need to define the distributivity maps

$$\begin{aligned} d_{\underline{u}, \underline{v}}^l &: (F \otimes_{\underline{u}, \underline{v}} G) \oplus (F' \otimes_{\underline{u}, \underline{v}} G) \rightarrow (F \oplus F') \otimes_{\underline{u}, \underline{v}} G, \\ d_{\underline{u}, \underline{v}}^r &: (F \otimes_{\underline{u}, \underline{v}} G) \oplus (F \otimes_{\underline{u}, \underline{v}} G') \rightarrow F \otimes_{\underline{u}, \underline{v}} (G \oplus G'). \end{aligned}$$

Given an object $\underline{f} \odot \underline{g}$ of $\mathcal{C}/\underline{u} \odot \underline{v}$ we define $(d_{\underline{u}, \underline{v}}^l)_{\underline{f} \odot \underline{g}}$ by the following diagram

$$\begin{array}{ccc} (F \otimes_{\underline{u}, \underline{v}} G)(\underline{f} \odot \underline{g}) \oplus (F' \otimes_{\underline{u}, \underline{v}} G)(\underline{f} \odot \underline{g}) & \xrightarrow{(d_{\underline{u}, \underline{v}}^l)_{\underline{f} \odot \underline{g}}} & ((F \oplus F') \otimes_{\underline{u}, \underline{v}} G)(\underline{f} \odot \underline{g}) \\ \lambda_{\underline{f}, \underline{g}}^{-1} \oplus \lambda_{\underline{f}, \underline{g}}^{-1} \downarrow & & \uparrow \lambda_{\underline{f}, \underline{g}} \oplus \lambda_{\underline{f}, \underline{g}} \\ F(\underline{f}) \otimes G(\underline{g}) \oplus F'(\underline{f}) \otimes G(\underline{g}) & \xrightarrow{d^l} & (F(\underline{f}) \oplus F'(\underline{f})) \otimes G(\underline{g}). \end{array}$$

We define $d_{\underline{u}, \underline{v}}^r$ in a similar fashion.

This way defined $\otimes_{\underline{u}, \underline{v}}$ satisfies conditions **(c.1)** – **(c.3)** as similar conditions are satisfied in \mathcal{D} . We also have that for any objects

$$F = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in} \quad \text{and} \quad G = \sum_{j=1}^s G_{j1} \otimes \cdots \otimes G_{jm}$$

of $\Psi(\underline{u})$ and $\Psi(\underline{v})$ respectively, we have

$$\begin{aligned} F \otimes_{\underline{u}, \underline{v}} \mathbb{O}_{\underline{v}} &= \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in} \otimes \mathbb{O}_{v_1} \otimes \cdots \otimes \mathbb{O}_{v_m} = \mathbb{O}_{\underline{u} \odot \underline{v}} \\ \mathbb{O}_{\underline{u}} \otimes_{\underline{u}, \underline{v}} G &= \sum_{j=1}^s \mathbb{O}_{u_1} \otimes \cdots \otimes \mathbb{O}_{u_n} \otimes G_{j1} \otimes \cdots \otimes G_{jm} = \mathbb{O}_{\underline{u} \odot \underline{v}} \end{aligned}$$

and thus condition **(c.4)** is satisfied.

We want to show next that if $(q, \underline{f}) : \underline{u} \rightarrow \underline{u}'$ and $(p, \underline{g}) : \underline{v} \rightarrow \underline{v}'$ are morphisms

in \mathcal{A} , then the following diagram is commutative

$$\begin{array}{ccc} \Psi(\underline{u}) \times \Psi(\underline{v}) & \xrightarrow{\Psi(\sigma, \underline{f}) \times \Psi(\rho, \underline{g})} & \Psi(\underline{u}') \times \Psi(\underline{v}') \\ \otimes \downarrow & & \downarrow \otimes \\ \underline{u}, \underline{v} & & \underline{u}', \underline{v}' \\ \Psi(\underline{u} \odot \underline{v}) & \xrightarrow{\Psi(\sigma \odot \rho, \underline{f} \odot \underline{g})} & \Psi(\underline{u}' \odot \underline{v}'), \end{array}$$

where $p \odot q : \underline{n} + \underline{m} \rightarrow \underline{n}' + \underline{m}'$ is the injective map that acts as q in the first n -elements and acts as p in the last m -elements. To show the commutativity of this diagrams suppose first that $q = \sigma \in \Sigma_n$ and $p = \rho \in \Sigma_m$ are permutations. Take

$$F = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in} \quad \text{and} \quad G = \sum_{j=1}^s G_{j1} \otimes \cdots \otimes G_{jm}$$

objects of $\Psi(\underline{u})$ and $\Psi(\underline{v})$ respectively. Then by definition we have

$$\begin{aligned} & \Psi(\sigma, \underline{f}) F \otimes_{\underline{u}', \underline{v}'} \Psi(\rho, \underline{g}) G \\ &= \left(\sum_{i=1}^r f_1^* F_{i\sigma^{-1}(1)} \otimes \cdots \otimes f_n^* F_{i\sigma^{-1}(n)} \right) \otimes_{\underline{u}', \underline{v}'} \left(\sum_{i=1}^r g_1^* G_{\rho^{-1}(1)} \otimes \cdots \otimes g_n^* G_{i\rho^{-1}(n)} \right) \\ &= \sum_{j=1}^s \sum_{i=1}^r \left(f_1^* F_{i\sigma^{-1}(1)} \otimes \cdots \otimes f_n^* F_{i\sigma^{-1}(n)} \otimes g_1^* G_{\rho^{-1}(1)} \otimes \cdots \otimes g_n^* G_{i\rho^{-1}(n)} \right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \Psi(\sigma \odot \rho, \underline{f} \odot \underline{g}) \left(F \otimes_{\underline{u}', \underline{v}'} G \right) \\ &= \Psi(\sigma \odot \rho, \underline{f} \odot \underline{g}) \left(\sum_{j=1}^s \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in} \otimes G_{j1} \otimes \cdots \otimes G_{jm} \right) \\ &= \sum_{j=1}^s \sum_{i=1}^r \left(f_1^* F_{i\sigma^{-1}(1)} \otimes \cdots \otimes f_n^* F_{i\sigma^{-1}(n)} \otimes g_1^* G_{\rho^{-1}(1)} \otimes \cdots \otimes g_n^* G_{i\rho^{-1}(n)} \right). \end{aligned}$$

This shows that condition **(c.5)** is satisfied. In addition, as each $\psi(\sigma, \underline{f})$ is strict map, and the distributivity maps are natural, this also shows that condition **(c.6)** is satisfied. A similar argument shows that the same is true for a general morphism in (q, \underline{f}) and (p, \underline{g}) in \mathcal{A} .

We now want to construct a multiplicative unit $\mathbf{1}$. This is an object in $\Psi(\underline{()})$. By definition, the objects of $\Psi(\underline{()})$ are the functors of the form $F = \sum_{i=1}^r F_i$ with F_i an object of $\text{Hom}_{\mathcal{C}}(\mathcal{C}/(), \mathcal{D})$. In particular, the functor $\mathbf{1} : \mathcal{C}/() \rightarrow \mathcal{D}$ that sends the only object of $\mathcal{C}/()$ to the unit 1 of \mathcal{D} is an object of $\Psi(\underline{()})$. We claim that this is a unit for $\otimes_{\underline{u}, \underline{v}}$. Indeed, if $F = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in}$ is an object in $\Psi(\underline{u})$, then

$$F \otimes_{\underline{u}, ()} \mathbf{1} = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes (F_{in} \otimes 1) = F$$

and similarly $\mathbf{1} \otimes_{(\cdot), \underline{v}} G = G$ for any object G of $\Psi(\underline{v})$. This shows that condition (c.7) is satisfied. On the other hand, from the coherences in the category \mathcal{D} we see at once that conditions (c.8) – (c.10) are also satisfied.

We now want to study the symmetry of the functor $\otimes_{\underline{u}, \underline{v}}$. Note that given objects $\underline{u} = (u_1, \dots, u_n)$ and $\underline{v} = (v_1, \dots, v_m)$ of \mathcal{A} , then the symmetry isomorphism $\tau_{\gamma, \underline{v}, \underline{u}} = \gamma^\odot : \underline{v} \odot \underline{u} \rightarrow \underline{u} \odot \underline{v}$ is given by the pair $(\xi, \underline{\text{id}})$, where $\xi \in \Sigma_{n+m}$ is the permutation that interchanges the first m -block with the last n -block and $\underline{\text{id}}$ is the $(n + m)$ -tuple of copies of id . With this in mind note that if

$$F = \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in} \quad \text{and} \quad G = \sum_{j=1}^s G_{j1} \otimes \cdots \otimes G_{jm},$$

are objects of $\Psi(\underline{u})$ and $\Psi(\underline{v})$ respectively, then we obtain

$$\begin{aligned} \Psi(\tau_{\gamma, \underline{v}, \underline{u}})(G \otimes_{\underline{v}, \underline{u}} F) &= \Psi(\tau_{\gamma, \underline{v}, \underline{u}}) \left(\sum_{j=1}^s \sum_{i=1}^r G_{j1} \otimes \cdots \otimes G_{jm} \otimes F_{i1} \otimes \cdots \otimes F_{in} \right) \\ &= \sum_{j=1}^s \sum_{i=1}^r F_{i1} \otimes \cdots \otimes F_{in} \otimes G_{j1} \otimes \cdots \otimes G_{jm} \end{aligned}$$

and

$$F \otimes_{\underline{u}, \underline{v}} G = \sum_{i=1}^r \sum_{j=1}^s F_{i1} \otimes \cdots \otimes F_{in} \otimes G_{j1} \otimes \cdots \otimes G_{jm}.$$

Define

$$\mu_{\underline{u}, \underline{v}} : F \otimes_{\underline{u}, \underline{v}} G \rightarrow \Psi(\tau_{\gamma, \underline{v}, \underline{u}})(G \otimes_{\underline{v}, \underline{u}} F)$$

to be the natural isomorphism that rearranges the summation order; that is, $\mu_{\underline{u}, \underline{v}}$ is an iterated application of the natural isomorphism γ^\oplus in \mathcal{D} . With this definition, it follows by the use of coherence theory that the conditions (c.11) – (c.14) of theorem 25 are satisfied. \square

Corollary 27 *A fibered bipermutative category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ gives rise to a multifunctor $T_1 : E\Sigma_* \rightarrow \mathbb{P}^{\mathcal{A}}$. By composing with the multifunctor K_* we obtain a multifunctor $T : E\Sigma_* \rightarrow \Sigma\mathcal{S}^{\mathcal{A}}$.*

6 Model Categories

In this section we use the machinery of closed model categories to rigidify a given multifunctor $T : E\Sigma_* \rightarrow \Sigma\mathcal{S}^{\mathcal{E}}$, as to obtain a multifunctor $T' : * \rightarrow \Sigma\mathcal{S}^{\mathcal{E}}$, here \mathcal{E} is a general permutative category. The application that we have in mind is the multifunctor $T : E\Sigma_* \rightarrow \Sigma\mathcal{S}^{\mathcal{A}}$ provided by the previous corollary. Here $*$ is the operad whose algebras are commutative monoids which can be

seen as a multicategory with only one object. Later we show that having a multifunctor $* \rightarrow \Sigma \mathcal{S}^{\mathcal{E}}$ is equivalent to having a lax map $\mathcal{E} \rightarrow \Sigma \mathcal{S}$. Thus in the case of the $T : E\Sigma_* \rightarrow \Sigma \mathcal{S}^A$ provided by the previous corollary, we obtain a lax map $\mathcal{A} \rightarrow \Sigma \mathcal{S}$. Finally, we show that this map induces a lax map $(\mathcal{C}^{op})^{-1}\mathcal{C}^{op} \rightarrow \Sigma \mathcal{S}$ via a right adjoint in a Quillen equivalence coming from a lax map $\mathcal{A} \rightarrow (\mathcal{C}^{op})^{-1}\mathcal{C}^{op}$.

The core of this section is to give a model structure to $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma \mathcal{S})$. This is done in (4) for the case that \mathcal{E} is the trivial category. In this case, the category $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma \mathcal{S})$ is isomorphic to the category of multifunctors $\mathcal{M} \rightarrow \Sigma \mathcal{S}$. The case where \mathcal{E} is a general permutative category follows by a straight forward modification of the construction in (4). We include the proofs here for completeness following very closely the treatment in (4, Section 11). Therefore no originality is claimed for this part of the work. The reader is suggested to read the beautiful work of Elmendorf and Mandell (4).

To begin, we recall briefly cofibrantly generated model categories and the positive model structure on symmetric spectra. For a complete discussion about cofibrantly generated model categories we refer the reader to (14) and (7) and for symmetric spectra (8) and (11).

Let λ an ordinal and \mathcal{C} a cocomplete category. We can see λ as a category with a unique arrow $f : x \rightarrow y$ whenever $x \leq y$. By a λ -sequence we mean a colimit preserving functor $X : \lambda \rightarrow \mathcal{C}$. We refer to the map $X_0 \rightarrow \text{colim}_{\beta < \lambda} X_\beta$ as the composition of the λ -sequence.

Definition 28 *Let I be a set of maps in a category \mathcal{C} that is cocomplete. A map $f : A \rightarrow B$ in \mathcal{C} is said to be a relative I -cell complex if there exists an ordinal λ and a λ -sequence $X : \lambda \rightarrow \mathcal{C}$ such that f is the composition of X and for each β with $\beta + 1 < \lambda$ there is a pushout*

$$\begin{array}{ccc} C_\beta & \longrightarrow & X_\beta \\ g_\beta \downarrow & & \downarrow \\ D_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

such that $g_\beta \in I$. The collection of relative I -cell complexes will be denoted by I -cell.

Thus a relative I -complex is a (possibly) transfinite composition of maps that are pushouts of maps in I .

Definition 29 *Let I be a class of maps in a category \mathcal{C} . A map f is said to be I -injective if it has the right lifting property with respect to any map in I . We denote by I -inj. the class of I -injective maps in \mathcal{C} . Similarly, a map f is said to be I -projective if it has the left lifting property with respect to any map in I . We denote by I -proj. the class of I -projective maps in \mathcal{C} .*

Definition 30 A model structure on a small category \mathcal{C} is said to be cofibrantly generated if there are sets I and J of maps such that the following conditions are satisfied.

- The domains of the maps of I are small relative to I -cell.
- The domains of the maps of J are small relative to J -cell.
- The class of fibrations is J -inj.
- The class of trivial fibrations is I -inj.

Let us review now briefly the positive closed model structure on symmetric spectra. This is an example of cofibrantly generated closed model structure with generating sets that we denote by I^+ and J^+ . The set I^+ can explicitly be described as follows. For $m \geq 0$, let F_m be the left adjoint functor to the evaluation functor Ev_m from symmetric spectra to simplicial sets, then

$$I^+ = \{F_m \partial \Delta[n]_+ \rightarrow F_m \Delta[n]_+ \mid m > 0, n \geq 0\}.$$

Note that the maps in I^+ have small domain and codomain. The set J^+ can be described in a similar fashion, all we need to know is that all the maps in J^+ also have small domain and codomain. According to this, the positive closed model structure on symmetric spectra has as weak equivalences the stable maps of symmetric spectra; that is, the maps $f : X \rightarrow Y$ such that $f^* : [Y, E] \rightarrow [X, E]$ is a bijection for all injective Ω -spectrum E . A map of symmetric spectra is a cofibration if and only if it is a retract of a relative I^+ -complex and a map is an acyclic cofibration if and only if it is a retract of a relative J^+ -complex. The fibrations are the maps that satisfy the right lifting property with respect to the maps in J^+ .

As mentioned before we want to rectify a multifunctor $T : E\Sigma_* \rightarrow \Sigma\mathcal{S}^\mathcal{E}$ as to obtain a multifunctor $T' : * \rightarrow \Sigma\mathcal{S}^\mathcal{E}$. As our first step, we give a closed model category structure to the category of multifunctors $E\Sigma_* \rightarrow \Sigma\mathcal{S}^\mathcal{E}$. To avoid any confusion we work in a general context. Thus let us fix \mathcal{M} a multicategory enriched over \mathbf{Cat} which we can see as enriched over simplicial sets by taking the nerve. We will give a model structure to the category $\mathbf{M}(\mathcal{M}, \Sigma\mathcal{S}^\mathcal{E})$ of multifunctors $\mathcal{M} \rightarrow \Sigma\mathcal{S}^\mathcal{E}$. As our first step, we identify the categories $\mathbf{M}(\mathcal{M}, \Sigma\mathcal{S}^\mathcal{E})$ and $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ under the canonical isomorphism that sends a multifunctor $F : \mathcal{M} \rightarrow \Sigma\mathcal{S}^\mathcal{E}$ to the assignment $F' : \mathcal{M} \times \mathcal{E} \rightarrow \Sigma\mathcal{S}$ defined by $F'(m, c) = F(m)(c)$. F' is a multifunctor in the first variable and a functor in the second variable and as explain before, this defines an isomorphism of categories. Next, we give a model structure to the category $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$.

Let us denote by $\mathcal{O}_\mathcal{M}$ (resp. $\mathcal{O}_\mathcal{E}$) the object set of \mathcal{M} (resp. \mathcal{E}). On the product category $\Sigma\mathcal{S}^{\mathcal{O}_\mathcal{M} \times \mathcal{O}_\mathcal{E}}$ we have a closed model structure for every closed model structure on $\Sigma\mathcal{S}$, in particular, we have a model category on $\Sigma\mathcal{S}^{\mathcal{O}_\mathcal{M} \times \mathcal{O}_\mathcal{E}}$ coming from the positive model structure on $\Sigma\mathcal{S}$. We will show that the category $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ can be seen as the category of algebras of a suitable monad

on $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$. Using this monad we will see that we can lift the positive model category on $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ to the category $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$. This way we will obtain a cofibrantly generated model category on $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$.

To begin, consider the functor $\iota_{(a,c)} : \Sigma\mathcal{S} \rightarrow \Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ for each object $(a, c) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}$, that is defined on objects as follows. Take X a symmetric spectrum, then

$$(\iota_{(a,c)}X)_{(b,d)} = \begin{cases} X & \text{if } (a, c) = (b, d) \\ * & \text{else.} \end{cases}$$

Similarly, $\iota_{(a,c)}$ is defined on morphisms. The functor $\iota_{(a,c)}$ is precisely the left adjoint to the projection functor $\pi_{(a,c)} : \Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}} \rightarrow \Sigma\mathcal{S}$. The positive stable model structure on $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ is cofibrantly generated with generating sets ιI^+ and ιJ^+ , where

$$\begin{aligned} \iota I^+ &= \{\iota_{(a,c)}f \mid f \in I^+, (a, c) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}\}, \\ \iota J^+ &= \{\iota_{(a,c)}f \mid f \in J^+, (a, c) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}\}. \end{aligned}$$

Thus a map in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ is a cofibration if and only if it is a retract of a relative ιI^+ -complex and a map in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ is an acyclic cofibration if and only if it is a retract of a relative ιJ^+ -complex.

Given $(a_i, d) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}$ for $1 \leq i \leq n$, we are going to use the notation

$$F_{(a_1, \dots, a_n; d)} = F_{(a_1, d)} \wedge \cdots \wedge F_{(a_n, d)}.$$

With this in mind we have the following definition.

Definition 31 For $(b, c) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}$ and $F : \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}} \rightarrow \Sigma\mathcal{S}$, let

$$(\mathbb{D}F)_{(b,c)} = \bigvee_{d \in \mathcal{O}_{\mathcal{E}}, \mathcal{E}(d,c)} \bigvee_{n \geq 0} \left(\bigvee_{a_1, \dots, a_n \in \mathcal{O}_{\mathcal{M}}} \mathcal{M}(a_1, \dots, a_n; b)_+ \wedge F_{(a_1, \dots, a_n; d)} \right) / \Sigma_n.$$

Let $\eta : F \rightarrow \mathbb{D}F$ be the map such that for $(b, c) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}$

$$\eta_{(b,c)} : F_{(b,c)} \cong 1_{b_+} \wedge F_{(b,c)} \rightarrow \mathcal{M}(b; b)_+ \wedge F_{(b,c)} \rightarrow (\mathbb{D}F)_{(b,c)}$$

and $\mu : \mathbb{D}\mathbb{D}F \rightarrow \mathbb{D}F$ is the map induced by the multiproduct in \mathcal{M} and the composition in \mathcal{E} .

By a direct and standard computation, it follows that (\mathbb{D}, μ, η) is a simplicial monad on the category $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$. Also, we see that a \mathbb{D} -algebra structure on an object of $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ is equivalent to an assignment $\mathcal{M} \times \mathcal{E} \rightarrow \Sigma\mathcal{S}$ that is a multifunctor in the first variable and a functor in the second variable and that the simplicial category of \mathbb{D} -algebras is isomorphic to $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$. Finally, if we see \mathbb{D} as a functor

$$\mathbb{D} : \Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}} \rightarrow \mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S}),$$

then \mathbb{D} is left adjoint to the forgetful functor $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S}) \rightarrow \Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$. From now on, if $a_1, \dots, a_n, x, b \in \mathcal{O}_{\mathcal{M}}$ and for $k \geq 0$, we will denote the $(n+k)$ -morphism set $\mathcal{M}(a_1, \dots, a_n, x, \dots, x, b)$ with k -copies of x by $\mathcal{M}(a_1, \dots, a_n, x^k, b)$. With this notation, note that in particular if $F = \iota_{(x,y)}X$, for some $(x, y) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}$ and a symmetric spectrum X , then for $(b, c) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}$

$$(\mathbb{D}\iota_{(x,y)}X)_{(b,c)} = \bigvee_{\mathcal{E}(y,c)} \bigvee_{n \geq 0} \mathcal{M}(x^n; b)_+ \wedge_{\Sigma_n} X^{(n)}$$

One advantage of seeing $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ as the category of \mathbb{D} -algebras, is the following lemma.

Lemma 32 *The category $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is bicomplete.*

Proof: Since $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is the category of algebras over a monad on a complete category, $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is complete. On the other hand, it follows as in (2, Proposition 7.2) that \mathbb{D} preserves reflexive coequalizers, thus by (2, Proposition 7.4) it follows that $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is cocomplete with colimits formed in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ as in (2, Proposition 7.4); that is, given a diagram F_i of \mathbb{D} -algebras, with structural maps $\xi_i : \mathbb{D}F_i \rightarrow F_i$, let $\text{colim}_i F_i$ be its colimit in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ and let $f_i : F_i \rightarrow \text{colim}_i F_i$ be the natural maps. Also consider

$$\alpha : \text{colim}_i \mathbb{D}F_i \rightarrow \mathbb{D}\text{colim}_i F_i$$

the unique map in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ whose composite is the natural map $\mathbb{D}F_i \rightarrow \text{colim}_i \mathbb{D}F_i$. Then the underlying object of the colimit of the F_i 's in the category $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is the (reflexive) coequalizer in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ of

$$\mathbb{D}(\text{colim}_i \mathbb{D}F_i) \rightrightarrows \mathbb{D}(\text{colim}_i F_i),$$

where one of the maps is $\mathbb{D}(\text{colim}_i \xi_i)$ and the other map is $\mu \circ \mathbb{D}\alpha$. \square

We are now ready to give the weak equivalences, fibrations and cofibrations for the positive stable model category in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$.

Definition 33

- A morphism in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is said to be a weak equivalence if the underlying morphism in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ is a stable equivalence.
- A morphism in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is said to be a positive stable fibration if the underlying morphism in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ is a positive stable fibration.
- A morphism in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is said to be a positive stable cofibration if it satisfies the left lifting property with respect to the acyclic stable fibrations.

In order to show that under these definitions we obtain a closed model structure on $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$, we will show that this description is precisely the

description of a cofibrantly generated model structure on $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ with generating sets $\mathbb{D}\iota I^+$ and $\mathbb{D}\iota J^+$, where

$$\begin{aligned}\mathbb{D}\iota I^+ &= \{\mathbb{D}\iota_{(a,c)}f \mid f \in I^+, (a, c) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}\}, \\ \mathbb{D}\iota J^+ &= \{\mathbb{D}\iota_{(a,c)}f \mid f \in J^+, (a, c) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}\}.\end{aligned}$$

To do so, we will first characterize the positive stable fibrations and acyclic positive fibrations in the proposition below.

Proposition 34

- A map in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is an acyclic positive fibration if and only if it has the right lifting property with respect to retracts of relative $\mathbb{D}\iota I^+$ -complexes; that is, the class of acyclic positive fibrations equals $\mathbb{D}\iota I^+$ -inj.
- A map in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is a positive fibration if and only if it has the right lifting property with respect to retracts of relative $\mathbb{D}\iota J^+$ -complexes; that is, the class of positive fibrations equals $\mathbb{D}\iota J^+$ -inj.

Proof: The result follows by the adjunction of \mathbb{D} and the forgetful functor. We show the first result, the second part follows in a similar fashion. As explained before a map in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ is a cofibration if and only if it is a retract of a relative ιI^+ -complex. Thus the positive stable acyclic fibrations on $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ are precisely the maps that satisfy the right lifting property with respect to retracts of relative ιI^+ -complexes. Thus by the adjunction of \mathbb{D} and the forgetful functor, a map in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ satisfies the right lifting property with respect retracts of relative $\mathbb{D}\iota I^+$ -complexes if and only if the underlying map in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ satisfies the right lifting property with respect to retracts of relative ιI^+ -complexes. \square

The hardest part of showing that under the given definitions of weak equivalences, fibrations and cofibrations we obtain a closed model structure on $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is the proposition below. Once we prove it, all the properties of a closed model structure are easy to verify and follow by applications of Quillen's small object argument and by the adjunction of \mathbb{D} and the forgetful functor.

Proposition 35 *A relative $\mathbb{D}\iota J^+$ -complex is a stable equivalence.*

To prove this proposition we follow the same ideas used to prove (4, Lemma 11.7); that is, we study pushouts in the category $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ of the form

$$F \coprod_{\mathbb{D}\iota_{(x,y)}X} \mathbb{D}\iota_{(x,y)}Y,$$

for some $(x, y) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}$, $f : X \rightarrow Y$ morphism in $\Sigma\mathcal{S}$ and $g : \iota_{(x,y)}X \rightarrow F$ morphism in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$.

We begin with the following lemma that helps us study certain coproducts in the category $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$.

Lemma 36 *Let \mathcal{C} be a cocomplete category and (\mathbb{T}, μ, η) a monad in \mathcal{C} that \mathbb{T} preserves reflexive coequalizers. Given a \mathbb{T} -algebra X with structural map $\xi_X : \mathbb{T}X \rightarrow X$ and an object Y of \mathcal{C} , the coproduct $X \amalg \mathbb{T}Y$ in the category of \mathbb{T} -algebras $\mathcal{C}[\mathbb{T}]$ is computed as the reflexive coequalizer in \mathcal{C}*

$$\mathbb{T}(\mathbb{T}X \amalg Y) \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} \mathbb{T}(X \amalg Y) \rightarrow P,$$

where

$$\begin{aligned} \alpha &= \mu \circ \mathbb{T}(\mathbb{T}(i_1 \amalg \mathbb{T}i_2 \circ \eta_Y)), \\ \beta &= \mathbb{T}(\xi_X \amalg i_2). \end{aligned}$$

Moreover, P is the coequalizer of the previous diagram in $\mathcal{C}[\mathbb{T}]$.

Proof: By (2, Lemma 6.6), we have that P is a \mathbb{T} -algebra and that P is the coequalizer in $\mathcal{C}[\mathbb{T}]$ of the diagram

$$\mathbb{T}(\mathbb{T}X \amalg Y) \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} \mathbb{T}(X \amalg Y) \rightarrow P.$$

As X is a \mathbb{T} -algebra, we see that X is the coequalizer of the diagram

$$\mathbb{T}\mathbb{T}X \begin{array}{c} \xrightarrow{\mu_X} \\ \xrightarrow{\mathbb{T}\xi_X} \end{array} \mathbb{T}X \rightarrow X.$$

Moreover the maps

$$\begin{aligned} h &= \mathbb{T}(i_{1\mathbb{T}X}) : \mathbb{T}\mathbb{T}X \rightarrow \mathbb{T}(\mathbb{T}X \amalg Y) \\ k &= \mathbb{T}(i_{1X}) : \mathbb{T}X \rightarrow \mathbb{T}(X \amalg Y) \end{aligned}$$

are so that the following diagrams are commutative

$$\begin{array}{ccc} \mathbb{T}\mathbb{T}X & \xrightarrow{\mu_X} & \mathbb{T}X \\ \downarrow h & & \downarrow k \\ \mathbb{T}(\mathbb{T}X \amalg Y) & \xrightarrow{\alpha} & \mathbb{T}(X \amalg Y), \end{array} \quad \begin{array}{ccc} \mathbb{T}\mathbb{T}X & \xrightarrow{\mathbb{T}\xi_X} & \mathbb{T}X \\ \downarrow h & & \downarrow k \\ \mathbb{T}(\mathbb{T}X \amalg Y) & \xrightarrow{\beta} & \mathbb{T}(X \amalg Y). \end{array}$$

By the universal property of the coequalizer, the previous diagrams guarantee the existence of a map $j_1 : X \rightarrow P$. Moreover, as both X and P are the coequalizers in the category of \mathbb{T} -algebras, we see that this map is a map of \mathbb{T} -algebras. On the other hand, we have the map $j_2 = \mathbb{T}(i_2) : \mathbb{T}Y \rightarrow \mathbb{T}(X \amalg Y)$ which is also a map of \mathbb{T} -algebras. By a direct computation one can see that

P together with the maps j_1 and j_2 satisfy the universal property of the coproduct in the category $\mathcal{C}[\mathbb{T}]$. \square

Note that we can apply the previous lemma to the monad \mathbb{D} , as $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ is bicomplete and \mathbb{D} preserves reflexive coequalizers. Thus if X is a symmetric spectrum, $(x, y) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}$ and F an object in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$, then the coproduct $F \amalg \mathbb{D}\iota_{(x,y)}X$ in the category $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is computed as the reflexive coequalizer in the category $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$

$$\mathbb{D}(\mathbb{D}F \amalg \iota_{(x,y)}X) \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} \mathbb{D}(F \amalg \iota_{(x,y)}X). \quad (8)$$

We can rewrite this coequalizer in a better way using the fact that the functor $-\wedge X$ preserves coequalizers. Thus define $T_0F = F$ and for $k \geq 1$, define T_kF in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ so that for $(b, c) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}$, $(T_kF)_{(b,c)}$ is the coequalizer in $\Sigma\mathcal{S}$ of the following diagram

$$\begin{array}{c} \bigvee_{\mathcal{E}(y,c)} \bigvee_{n \geq 0} \left(\bigvee_{a_1, \dots, a_n \in \mathcal{O}_{\mathcal{M}}} \mathcal{M}(a_1, \dots, a_n, x^k; b)_+ \wedge \mathbb{D}F_{(a_1, \dots, a_n; y)} \right) / \Sigma_n \\ \Rightarrow \bigvee_{\mathcal{E}(y,c)} \bigvee_{n \geq 0} \left(\bigvee_{a_1, \dots, a_n \in \mathcal{O}_{\mathcal{M}}} \mathcal{M}(a_1, \dots, a_n, x^k; b)_+ \wedge F_{(a_1, \dots, a_n; y)} \right) / \Sigma_n, \end{array}$$

where one of the maps is induced by the algebra structural map $\xi_F : \mathbb{D}F \rightarrow F$ and the other is induced by the multiproduct in \mathcal{M} and composition in \mathcal{E} . Using the T_kF 's, we can rewrite the coequalizer (8) as to obtain that for a symmetric spectrum X , $(x, y) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}$ and F an object in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$, the coproduct $F \amalg \mathbb{D}\iota_{(x,y)}X$ in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is given by

$$(F \amalg \mathbb{D}\iota_{(x,y)}X)_{(b,c)} = \bigvee_{k \geq 0} (T_kF)_{(b,c)} \wedge_{\Sigma_k} X^k.$$

We also use the T_k 's as defined above to study pushouts of the form

$$F \amalg_{\mathbb{D}\iota_{(x,y)}X} \mathbb{D}\iota_{(x,y)}X.$$

To do so, we will use the same construction $Q_i^k(g)$, for $k \geq 0$, $0 \leq i \leq k$ and a map of symmetric spectra $g : X \rightarrow Y$ as in (4, Section 12). We include it for completeness. The $Q_i^k(g)$'s are inductively defined as follows: $Q_0^k(g) = X^k$, $Q_k^k(g) = Y^k$. For $0 < i < k$, define $Q_i^k(g)$ as the pushout square

$$\begin{array}{ccc} \Sigma_{k+} \wedge_{\Sigma_{k-i} \times \Sigma_i} X^{k-i} \wedge Q_{i-1}^i(g) & \longrightarrow & \Sigma_{k+} \wedge_{\Sigma_{k-i} \times \Sigma_i} X^{k-i} \wedge Y^i \\ \downarrow & & \downarrow \\ Q_{i-1}^k(g) & \longrightarrow & Q_i^k(g) \end{array}$$

The Q_i^k 's are defined so that essentially, Q_i^k is the Σ_k -subspectrum of Y^k with i factors of Y and $k - i$ factors of X . We are now ready to describe our final construction. For an object F of $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$, let $F_0 = F$ and for $k > 0$ let F_k to be the pushout in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$

$$\begin{array}{ccc} T_k F \wedge_{\Sigma_k} Q_{k-1}^k(g) & \longrightarrow & T_k F \wedge_{\Sigma_k} Y^k \\ \downarrow & & \downarrow \\ F_{k-1} & \longrightarrow & F_k. \end{array}$$

Where the map in the top is the map induced by $Q_{k-1}^k(g) \rightarrow Y^k$ and the map on the left is induced by $\iota_{(x,y)X} \rightarrow F$. Let $F_\infty = \text{colim} F_k$ where the colimit is computed on $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$. The relevance of the F_k 's is given in the following proposition which corresponds to (4, Proposition 12.6).

Proposition 37 *F_∞ is isomorphic to the underlying object in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ of the pushout $\coprod_{\mathbb{D}\iota_{(b,c)X}} \mathbb{D}\iota_{(b,c)}Y$.*

Proof: The proof goes by showing that F_∞ satisfies the universal property of the pushout

$$F \coprod_{\mathbb{D}\iota_{(b,c)X}} \mathbb{D}\iota_{(b,c)}Y$$

in the category $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$. \square

With the given filtration $F = F_0 \rightarrow F_1 \rightarrow \cdots \rightarrow F_\infty$ of $F \coprod_{\mathbb{D}\iota_{(b,c)X}} \mathbb{D}\iota_{(b,c)}Y$ we can prove proposition 35.

Proof of proposition 35. We want to show that if $h : X \rightarrow Y$ is a relative $\mathbb{D}\iota J^+$ -complex then h is a positive stable equivalence. We are going to consider first the special case where h is obtained as a pushout $h : F \rightarrow F \coprod_{\mathbb{D}\iota_{(x,y)X}} \mathbb{D}\iota_{(x,y)}Y$, for a map $f : X \rightarrow Y$ of symmetric spectra in J^+ and $g : \iota_{(x,y)X} \rightarrow F$ a morphism in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$. We have a filtration $F_0 = F \rightarrow F_1 \rightarrow \cdots \rightarrow F_\infty$ as in the previous proposition such that $h : F = F_0 \rightarrow F_\infty$ is the transfinite composition of the maps $F_k \rightarrow F_{k+1}$. To prove that h is a positive stable equivalence it's enough to prove then that each map $F_k \rightarrow F_{k+1}$ is a positive stable equivalence as the composition of (transfinite) acyclic positive cofibrations is again an acyclic positive cofibration in the positive stable model category. By (4, Proposition 12.6) we have that $F_k \rightarrow F_{k+1}$ is an objectwise level cofibration of symmetric spectra. Also note that the quotient F_{k+1}/F_k is naturally isomorphic to $T_{k+1}F \wedge_{\Sigma_{k+1}} (Y/X)^{k+1}$. As $X \rightarrow Y$ is an acyclic positive cofibration it follows that Y/X is positive cofibrant and stably equivalent to $*$. Hence F_{k+1}/F_k is stably equivalent to $*$ in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$. But as mentioned before $F_k \rightarrow F_{k+1}$ is an objectwise level cofibration and then we conclude that $F_k \rightarrow F_{k+1}$ is a stable equivalence. To finish the proof, note that we can write any relative $\mathbb{D}\iota J^+$ -complex as a retract of a possible transfinite composition

of pushouts of the form we just considered. These are stable equivalences by the previous consideration. On the positive model structure a transfinite composition of stable equivalences is also a stable equivalence. This finishes the proof. \square

With the proof of the Proposition 35 completed we can now move on to proving that we in fact have defined a model structure on $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$. We start by constructing factorizations of morphisms in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$. To do so, note that as mentioned before, the domains of the maps in I^+ and J^+ are small, this implies that the domains of the maps in $\mathbb{D}I^+, \mathbb{D}J^+$ are also small. Therefore we can apply Quillen's small object argument as in (7, Theorem 2.1.14) to factorize a morphism f in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ in the form $f = pi$, where

- i is in $\mathbb{D}I^+$ -cell and p is in $\mathbb{D}I^+$ -inj. or
- i is in $\mathbb{D}J^+$ -cell and p is in $\mathbb{D}J^+$ -inj.

By Proposition 34 a map in $\mathbb{D}I^+$ -inj is precisely an acyclic positive fibration and a map in $\mathbb{D}J^+$ -inj is a positive fibration. Thus we obtain the following proposition.

Proposition 38

- A map f in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ can be factored as $f = pi$, where i is a relative $\mathbb{D}I^+$ -complex and p is an acyclic positive stable fibration.
- A map f in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ can be factored as $f = pi$, where i is a $\mathbb{D}J^+$ -complex and p is a positive stable fibration.

In order for these to be the required factorizations we need to prove the following lemma.

Lemma 39

- A map in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is a positive stable cofibration if and only if it is a retract of a relative $\mathbb{D}I^+$ -complex.
- A map in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is an acyclic positive stable cofibration if and only if it is a retract of a relative $\mathbb{D}J^+$ -complex.

Proof: Let's prove the first assertion. Any retract of a relative $\mathbb{D}I^+$ -complex satisfies the left lifting property with respect to the class $\mathbb{D}I^+$ -inj. Thus by Proposition 34 and the definition any retract of a relative $\mathbb{D}I^+$ -complex is a positive stable cofibration. On the other hand, take f a map that is a positive stable cofibration. By the Proposition 38, we can factor $f = pi$, where i is a relative $\mathbb{D}I^+$ -complex followed and p is an acyclic positive stable fibration. But since f is a cofibration, by definition it follows that f satisfies the left lifting property with respect to p . By (7, Lemma 1.19) we have that f is a

retract of i and thus f is a retract of a relative $\mathbb{D}L^+$ -complex.

The second assertion is proved in the same way. If f is a retract of a relative $\mathbb{D}LJ^+$ -complex then by Proposition 35 we have that f is a stable equivalence. Also, f satisfies the left lifting property with respect to the class $\mathbb{D}LJ^+$ -inj. and thus by Proposition 34, f satisfies the left lifting property with respect to the class of positive stable fibrations, in particular f is a positive stable cofibration. On the other hand, take f a map that is an acyclic stable cofibration. By the Proposition 38, we can factor $f = pi$, where i is a relative $\mathbb{D}LJ^+$ -complex followed and p is a positive stable fibration. By Proposition 35, i is a stable equivalence and so is f . Since stable equivalences satisfy the two out of three property we see that p is also a stable equivalence, hence p is an acyclic stable fibration. In particular f satisfies the left lifting property with respect to p and by (7, Lemma 1.19) we have that f is a retract of i and thus f is a retract of a relative $\mathbb{D}LJ^+$ -complex. \square

Combining Propositions 38 and 39 we obtain the desired factorizations as in the following lemma.

Lemma 40 *A map f in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ can be factored as $f = pi$, where i is a stable cofibration and p is an acyclic positive stable fibration or we can take i is an acyclic cofibration and p is a positive stable fibration.*

After proving these factorizations we are now ready to show that this choice of weak equivalences, fibrations and cofibrations gives rise a closed model category structure on the category $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$.

Theorem 41 *There exists a model category structure on $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ whose weak equivalences are the morphisms that are stable equivalences, the fibrations are the positive stable fibrations and the cofibrations are the positive stable cofibrations.*

Proof: We need to verify axioms MC1.-MC5. of (1, Definition 3.3). We need to note first that it is clear by the definition that the classes of weak equivalences, fibrations and cofibrations contain the identity and are closed under composition.

- MC1. By Proposition 32 we know that $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ has all small limits and colimits in particular those that are finite.
- MC2. Since in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ the two out of three property is satisfied, it follows at once that it is satisfied on $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$
- MC3. By the same reason as in MC2. retracts of weak equivalences are weak equivalences. Also by the characterizations of Propositions 34, 39 we see that fibrations and cofibrations are closed under retracts.
- MC4. The cofibrations satisfy the left lifting property with respect to acyclic

fibrations by definition. On the other hand by Propositions 34, 39 we have that fibrations satisfy the right lifting property with respect to acyclic fibrations.

- MC5. The factorization properties follow by Lemma 40.

□

Our next step toward rectifying a multifunctor $T : E\Sigma_* \rightarrow \Sigma\mathcal{S}^\mathcal{E}$ is to study the relationships between the previous model structure when we change the multicategory \mathcal{M} . To be more precise, we want to show that if $f : \mathcal{M} \rightarrow \mathcal{M}'$ is a multifunctor between multicategories enriched over simplicial sets, then

$$g^* := (f \times id)^* : \mathbf{M}_1\mathbf{F}_2(\mathcal{M}', \mathcal{E}; \Sigma\mathcal{S}) \rightarrow \mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$$

is a right adjoint in a Quillen adjunction. Also we want to show that in the case that f is a weak equivalence between multicategories, then this Quillen adjunction is actually a Quillen equivalence. We will apply this to the particular case where $\mathcal{M} = E\Sigma_*$ and $\mathcal{M}' = *$ as to get the desired rectification of the multifunctor $T : E\Sigma_* \rightarrow \Sigma\mathcal{S}^\mathcal{E}$. Once again we will follow very closely those ideas presented on (4) with only slight modifications as to adapt them to our situation.

Suppose we have a multifunctor $f : \mathcal{M} \rightarrow \mathcal{M}'$ between two small multicategories, let \mathcal{O} and \mathcal{O}' be the object sets of \mathcal{M} and \mathcal{M}' . Also let \mathbb{D} and \mathbb{D}' be the monads on $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ and $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}'} \times \mathcal{O}_{\mathcal{E}}}$, induced by \mathcal{M} and \mathcal{M}' respectively. By composing with $g := f \times id$, we obtain a functor between the product categories

$$\pi_g : \Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}'} \times \mathcal{O}_{\mathcal{E}}} \rightarrow \Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}.$$

This functor has a left adjoint which we denote by κ_g ,

$$\kappa_g : \Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}} \rightarrow \Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}'} \times \mathcal{O}_{\mathcal{E}}}.$$

An explicit formula for κ_g is given as follows: if $F : \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}} \rightarrow \Sigma\mathcal{S}$, then

$$(\kappa_g F)_{(b,c)} = \bigvee_{a \in f^{-1}(b)} F_{(a,c)}.$$

Note that we can see the multifunctor f as a natural transformation

$$\kappa_g \mathbb{D} \rightarrow \mathbb{D}' \kappa_g.$$

With this definition we can prove the following theorem.

Theorem 42 *Let \mathcal{M} and \mathcal{M}' be multicategories enriched over simplicial sets and $f : \mathcal{M} \rightarrow \mathcal{M}'$. Then there exists a functor*

$$g_{\#} : \mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S}) \rightarrow \mathbf{M}_1\mathbf{F}_2(\mathcal{M}', \mathcal{E}; \Sigma\mathcal{S})$$

left adjoint to g^* such that the pair $(g_{\#}, g^*)$ forms a Quillen adjunction.

Proof: For an object F of $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ define $g_{\#}F$ as the reflexive coequalizer of the following diagram

$$\mathbb{D}'\kappa_g\mathbb{D}F \rightrightarrows \mathbb{D}'\kappa_gF \rightarrow g_{\#}F,$$

where one of the arrows comes from the multifunctor f seen as a natural transformation $\kappa_g\mathbb{D} \rightarrow \mathbb{D}'\kappa_g$ and the other arrow comes from the algebra structure map $\xi F : \mathbb{D}F \rightarrow F$ of F . It follows at once by the universal property of this coequalizer and the adjunctions that this way defined $g_{\#}$ is left adjoint to

$$g^* : \mathbf{M}_1\mathbf{F}_2(\mathcal{M}', \mathcal{E}; \Sigma\mathcal{S}) \rightarrow \mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S}).$$

On the other hand, it follows by the definition that g^* preserves fibrations and acyclic fibrations. Thus by (1, Theorem 9.7) we get that the pair $(g_{\#}, g^*)$ forms a Quillen adjunction. \square

We will see now that in the case that f is a weak equivalences of multicategories then this Quillen adjunction actually forms a Quillen equivalence. We begin by recalling the definition of a weak equivalence between multicategories from (4, Section 12).

Definition 43 *Let \mathcal{M} and \mathcal{M}' be two multicategories enriched over simplicial sets and $f : \mathcal{M} \rightarrow \mathcal{M}'$ an (enriched) multifunctor. We say that f is a weak equivalence whenever the functor $\pi_0 f$ is an equivalence between the category of components and for all objects a_1, \dots, a_n, b of \mathcal{M} , the map of simplicial sets*

$$\mathcal{M}(a_1, \dots, a_n, b) \rightarrow \mathcal{M}'(f(a_1), \dots, f(a_n), f(b))$$

is a weak equivalence.

We show now that in the case where $f : \mathcal{M} \rightarrow \mathcal{M}'$ is a weak equivalence then the pair $(g_{\#}, g^*)$ forms a Quillen equivalence.

Theorem 44 *If $f : \mathcal{M} \rightarrow \mathcal{M}'$ is a weak equivalence between multicategories enriched over simplicial sets, then the Quillen adjunction $(g_{\#}, g^*,)$ forms a Quillen equivalence.*

Proof: By (1, Theorem 9.7), we need to show that for every cofibrant object A of $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ and every fibrant object B of $\Sigma\mathcal{S}^{\mathcal{M}' \times \mathcal{E}}$, a map $h : g_{\#}A \rightarrow B$ is a weak equivalence if and only if its adjoint $j : A \rightarrow g^*B$ is a weak equivalence. By the Lemma 45 below, we know that h is a weak equivalence if and only if g^*h is a weak equivalence. By definition, j equals the composite

$$A \xrightarrow{\psi} g^*g_{\#}A \xrightarrow{g^*h} g^*B$$

where $\psi : A \rightarrow g^* g_{\#} A$ is the unit of the adjunction. We show below in Theorem 47 that whenever A is cofibrant ψ is positive stable equivalence. By the two out of three axiom we see that $h : A \rightarrow g^* B$ is a weak equivalence if and only if its adjoint $j : g_{\#} A \rightarrow B$ is a weak equivalence \square

Lemma 45 *A map $h : F \rightarrow G$ in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}', \mathcal{E}; \Sigma\mathcal{S})$ is a weak equivalence if and only if $g^* h$ is a weak equivalence in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$.*

Proof: The proof is a straight forward generalization of the proof of (4, Lemma12.4). \square

Note that the initial object of $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ is the assignment

$$\begin{aligned} \mathcal{M} \times \mathcal{E} &\rightarrow \Sigma\mathcal{S} \\ (b, c) &\rightarrow \mathcal{M}(\cdot; b)_+ \wedge S. \end{aligned}$$

Definition 46 *Let λ an ordinal and F a cofibrant object of $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$. We say that F can be built in λ -stages, if we can find a λ -sequence*

$$X : \lambda \rightarrow \mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$$

such that $X_0 = \mathcal{M}(\cdot; -)_+ \wedge S$, the initial object, and for each β with $\beta + 1 < \lambda$ there is a pushout

$$\begin{array}{ccc} C_{\beta} & \longrightarrow & X_{\beta} \\ g_{\beta} \downarrow & & \downarrow \\ D_{\beta} & \longrightarrow & X_{\beta+1}, \end{array}$$

such that g_{β} is the coproduct of maps in $\mathbb{D}I^+$ and that $X_0 \rightarrow F$ is isomorphic to the transfinite composition $X_0 \rightarrow \text{colim}_{\beta < \lambda} X_{\beta}$.

Note that if $\beta \leq \lambda$ and A can be built in β -stages, then clearly A can be built in λ -stages. We are now ready to prove the following theorem.

Theorem 47 *If A is a cofibrant object in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$, then the unit*

$$\psi : A \rightarrow g^* g_{\#} A$$

of the Quillen adjunction $(g_{\#}, g^)$ is a stable equivalence.*

Proof: The proof of this theorem is taken from the proof of (4, Theorem 12.5) with small modifications as to fit into our situation. Let A be a cofibrant object in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$. Then $\mathcal{M}(\cdot; -)_+ \wedge S \rightarrow A$ is a retract of a relative $\mathbb{D}I^+$ -complex. It suffices then to prove the statement for the case where A can be built in λ -stages for an ordinal λ . The proof goes by transfinite induction on the stages that A can be built in. Thus we want to show that for every

ordinal λ and every cofibrant object that can be build in λ -stages, the map

$$\psi : A \rightarrow g^*g_{\#}A$$

is a stable equivalence. For $\lambda = 0$, we have that a cofibrant object that can be build in 0-stages is isomorphic to $\mathcal{M}(-)_+ \wedge S$. In this case

$$\begin{aligned} g^*g_{\#}(\mathcal{M}(-)_+ \wedge S) &= \mathcal{M}'(-)_+ \wedge S, \\ A = \mathcal{M}(-)_+ \wedge S &\rightarrow \mathcal{M}'(-)_+ \wedge S = g^*g_{\#}A \end{aligned}$$

and the statement follows as f is weak equivalence.

We also need the case $\lambda = 1$. In this case, a cofibrant object that can be built in 1-stage is an object of the form $A = \mathbb{D}F$, where F is an object in $\Sigma\mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ that is objectwise cofibrant. In this situation the map $A \rightarrow g^*g_{\#}A$ equals

$$\mathbb{D}X \rightarrow g^*\mathbb{D}\kappa_g X.$$

By definition we see that for $(b, c) \in \mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}$

$$(g^*\mathbb{D}\kappa_g X)_{(b,c)} = \bigvee_{d \in \mathcal{O}_{\mathcal{E}}, \mathcal{E}(d,c), n \geq 0} \left(\bigvee_{a'_1, \dots, a'_n \in \mathcal{O}_{\mathcal{M}'}} \bigvee_{i=1}^n \bigvee_{a_i \in f^{-1}(a'_i)} (\mathcal{M}'_{a'_1, \dots, a'_n; f(b)})_+ \wedge F_{(a_1, \dots, a_n; d)} \right) / \Sigma_n.$$

Where here $(\mathcal{M}'_{a'_1, \dots, a'_n; f(b)})$ means $\mathcal{M}'(a'_1, \dots, a'_n; f(b))$. Since f is a weak equivalence, it follows that $\mathbb{D}X \rightarrow g^*\mathbb{D}\kappa_g X$ is a stable equivalence and thus the statement is true for $\lambda = 1$. Suppose the statement is true for all ordinals $\beta < \lambda$ and that λ is a limit ordinal. Take a λ -sequence $X : \lambda \rightarrow \mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ with $X_0 = \mathcal{M}(-)_+ \wedge S$ and with $\mathcal{M}(-)_+ \wedge S \rightarrow \text{colim}_{\beta < \lambda} X_{\beta}$ isomorphic to $X_0 \rightarrow A$. We have that

$$g^*g_{\#}X_{\beta} = \text{colim}_{\beta < \lambda} g^*g_{\#}X_{\beta} \quad (9)$$

and since each X_{β} can be build in β -stages and $\beta < \lambda$, the statement is true for $X_0 \rightarrow X_{\beta}$ and by (9) we see that the statement is true for X_{λ} and hence for A . Thus we are left to prove the statement for $\beta + 1$ provided it's true for β and $\beta \geq 1$. By a separate transfinite induction, this case reduces to showing that if A is a cofibrant object with $A \rightarrow g^*g_{\#}A$ a stable equivalence then the same is true for B , where B is obtained as the pushout in $\mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma\mathcal{S})$ of the following diagram

$$\begin{array}{ccc} \mathbb{D}\iota_{(x,y)}X & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathbb{D}\iota_{(x,y)}Y & \longrightarrow & B. \end{array}$$

By Proposition 37, we have filtration $A = A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_{\infty} = B$, where each map $A_k \rightarrow A_{k+1}$ is an objectwise level cofibration. The associated graded

to this filtration is

$$\bigvee_{k \geq 0} T_k \wedge_{\Sigma_k} (Y/X)^k.$$

Note that this is isomorphic in $\Sigma \mathcal{S}^{\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{E}}}$ to the underlying object of

$$A \coprod \mathbb{D}_{\iota(x,y)}(Y/X).$$

Let $A' = g_{\#}A$ and $B' = g_{\#}B$. Note that

$$B' = A' \coprod_{\mathbb{D}'\iota(f(x),y)-X} \mathbb{D}'\iota(f(x),y)X.$$

For B' we also have a filtration $A' = A'_0 \rightarrow A'_1 \rightarrow \cdots \rightarrow A'_\infty = B'$, where each map $A'_k \rightarrow A'_{k+1}$ is an objectwise level cofibration and whose associated graded is isomorphic in $\Sigma \mathcal{S}^{\mathcal{O}_{\mathcal{M}'} \times \mathcal{O}_{\mathcal{E}'}}$ to $A' \coprod \mathbb{D}'\iota(f(x),y)(Y/X)$. The map $B \rightarrow g^*B' = \pi_g B'$ preserves the given filtrations. Also the map of associated graded is

$$A \coprod \mathbb{D}_{\iota(x,y)}(Y/X) \rightarrow \pi_g(A' \coprod \mathbb{D}'\iota(f(x),y)(Y/X)) \approx g^*g_{\#}(A \coprod \mathbb{D}_{\iota(x,y)}(Y/X)),$$

and we that this is a stable equivalence. This follows from the fact that the cofibrant object $A \coprod \mathbb{D}_{\iota(x,y)}(Y/X)$ can be built in β stages. Indeed, suppose $W : \beta \rightarrow \mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma \mathcal{S})$ is a β -sequence such that $W_0 = \mathcal{M}(-)_+ \wedge S$ and for each γ with $\gamma + 1 < \beta$ there is a pushout

$$\begin{array}{ccc} C_\gamma & \longrightarrow & W_\gamma \\ g_\gamma \downarrow & & \downarrow \\ D_\gamma & \longrightarrow & W_{\gamma+1} \end{array}$$

such that g_γ is the coproduct of maps in $\mathbb{D}\iota I^+$ and that $W_0 \rightarrow A$ is the transfinite composition $W_0 \rightarrow \operatorname{colim}_{\gamma < \beta} W_\gamma$. Let $W' : \beta \rightarrow \mathbf{M}_1\mathbf{F}_2(\mathcal{M}, \mathcal{E}; \Sigma \mathcal{S})$ be the β -sequence defined as follows: Take $W'_0 = \mathcal{M}(-)_+ \wedge S$, define W'_1 as the pushout of the diagram

$$\begin{array}{ccc} C_0 \coprod \mathcal{M}(-)_+ \wedge S & \longrightarrow & W'_0 \\ g_0 \coprod i \downarrow & & \downarrow \\ D_0 \coprod \mathbb{D}_{\iota(x,y)}(Y/X) & \longrightarrow & W'_1. \end{array}$$

Thus $W'_1 = W_1 \coprod \mathbb{D}_{\iota(x,y)}(Y/X)$. In general define for $\gamma \geq 2$

$$W'_\gamma = W_\gamma \coprod \mathbb{D}_{\iota(x,y)}(Y/X).$$

This way defined we see that for all $\gamma < \beta$, $W'_{\gamma+1}$ is the pushout of

$$\begin{array}{ccc} C_\gamma & \longrightarrow & W'_\gamma \\ g_\gamma \downarrow & & \downarrow \\ D_\gamma & \longrightarrow & W'_{\gamma+1}. \end{array}$$

By the induction hypothesis, it follows that

$$A \coprod \mathbb{D}l_{(x,y)}(Y/X) \rightarrow g^* g_\#(A \coprod \mathbb{D}l_{(x,y)}(Y/X))$$

is a stable equivalence. Since each map $A_k \rightarrow A_{k+1}$, and $A'_k \rightarrow A_{k+1}$ are objectwise level cofibrations and the map of associated graded is a stable equivalence we have that each map $A_k \rightarrow \pi_g A_k$ is a stable equivalence. From here we conclude that $B \rightarrow \pi_g B' \approx g^* g_\# B$ is a stable equivalence. \square

Corollary 48 *A multifunctor $T : E\Sigma_* \rightarrow \Sigma\mathcal{S}^\mathcal{E}$ gives rise to a multifunctor $T' : * \rightarrow \Sigma\mathcal{S}^\mathcal{E}$*

Proof: Consider the multifunctor $f : E\Sigma_* \rightarrow *$ that sends every k -morphism in $E\Sigma_k$ to the only k -morphism in $*$. This defines trivially a multifunctor between multicategories enriched over simplicial sets. (We see $E\Sigma_*$ enriched over simplicial sets by taking the nerve of the category $E\Sigma_k$, for $k \geq 0$). Note that for every $k \geq 0$, the geometric realization of $E\Sigma_k$ is contractible and thus f is a weak equivalence between the multicategories $\mathcal{M} = E\Sigma_*$ and $\mathcal{M} = *$. By Theorem 44, we have that the functor $g_\# := (f \times \text{id})_\#$ is a left adjoint in a Quillen equivalence. Let $T' = g_\#(T)$, then $T' : * \rightarrow \Sigma\mathcal{S}^\mathcal{E}$ is the required multifunctor. \square

We show in the following proposition that such a multifunctor structure is equivalent as having a lax map $F : \mathcal{E} \rightarrow \Sigma\mathcal{S}$.

Proposition 49 *Having a multifunctor $L : * \rightarrow \Sigma\mathcal{S}^\mathcal{E}$ is equivalent as having a lax map $F : \mathcal{E} \rightarrow \Sigma\mathcal{S}$*

Proof: Suppose we have a multifunctor $L : * \rightarrow \Sigma\mathcal{S}^\mathcal{E}$. The multicategory $*$ has only one object so we consider $F : \mathcal{E} \rightarrow \Sigma\mathcal{S}$ the image under L of this object. Thus $F : \mathcal{E} \rightarrow \Sigma\mathcal{S}$ is a functor. On $*$ we have a unique 2-morphism, let $\phi \in \Sigma\mathcal{S}^\mathcal{E}(F, F; F)$, be the image of this morphism. Unraveling the definitions we see that ϕ is a 2-linear natural transformation, thus for every objects c_1, c_2 of \mathcal{E} we have a map of symmetric spectra

$$\phi_{c_1, c_2} : F(c_1) \wedge F(c_2) \rightarrow F(c_1 \otimes c_2).$$

The collection of maps $\{\phi_{c_1, c_2}\}$ satisfy the naturality condition as in Definition 20. Suppose that $f : c_1 \rightarrow c'_1$, $g : c_2 \rightarrow c'_2$ are morphisms in \mathcal{E} . Then the

following naturality condition must be satisfied

$$\begin{array}{ccc}
F(c_1) \wedge F(c_2) & \xrightarrow{F(f) \wedge F(g)} & F(c'_1) \wedge F(c'_2) \\
\phi_{c_1, c_2} \downarrow & & \downarrow \phi_{c'_1, c'_2} \\
F(c_1 \otimes c_2) & \xrightarrow{F(f \otimes g)} & F(c'_1 \otimes c'_2)
\end{array}$$

This means that the map $\phi_{c_1, c_2} : F(c_1) \wedge F(c_2) \rightarrow F(c_1 \otimes c_2)$ is a natural map. On the other hand, $*$ has a unique 0-morphism, let $\eta \in \Sigma \mathcal{S}^{\mathcal{E}}$ be the image of this unique 0-morphism. Then $\eta : S \rightarrow F(1)$, where here S is the symmetric sphere spectrum. The maps η, ϕ satisfy some coherences coming from the fact that L preserves the multifunctor structure, for example if $\sigma \in \Sigma_2$ is the nontrivial element, then $\phi = \sigma^* \phi$, this means that for every objects c_1, c_2 in \mathcal{E} , the following diagram is commutative

$$\begin{array}{ccc}
F(c_1) \wedge F(c_2) & \xrightarrow{\alpha} & F(c_2) \wedge F(c_1) \\
\phi_{c_1, c_2} \downarrow & & \downarrow \phi_{c_2, c_1} \\
F(c_1 \otimes c_2) & \xrightarrow{F(\gamma)} & F(c_2 \otimes c_1)
\end{array}$$

here α is the natural isomorphism in symmetric spectra that changes the factors and γ is the natural isomorphism coming from the permutative structure on \mathcal{E} . The other coherences for a lax map are obtained in a similar way.

Conversely, suppose that $F : \mathcal{E} \rightarrow \Sigma \mathcal{S}$ is a lax map. Thus we have a natural maps $\eta : S \rightarrow F(1)$ and $\phi_{c_1, c_2} : F(c_1) \wedge F(c_2) \rightarrow F(c_1 \otimes c_2)$. Then using the coherences that these maps satisfy we see that the assignment

$$L : * \rightarrow \Sigma \mathcal{S}^{\mathcal{E}}$$

that sends the unique object of $*$ to F , the unique 0-morphism to η and the unique 2-morphism to ϕ defines a multifunctor. \square

According to Corollary 48 and Proposition 49, a multifunctor $T : E\Sigma_* \rightarrow \Sigma \mathcal{S}^{\mathcal{E}}$ can be rectified as to obtain a lax map $\vartheta : \mathcal{E} \rightarrow \Sigma \mathcal{S}$. In particular, the multifunctor $T : E\Sigma_* \rightarrow \Sigma \mathcal{S}^{\mathcal{A}}$ induced by a bipermutative category of fibers $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$, induces a lax map $\vartheta : \mathcal{A} \rightarrow \Sigma \mathcal{S}$. We show now that this lax map induces a lax map $\phi : \mathcal{C}^{op} \rightarrow \Sigma \mathcal{S}$. We show this in the following corollary.

Corollary 50 *A fibered bipermutative category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ induces a lax map $\phi : \mathcal{C}^{op} \rightarrow \Sigma \mathcal{S}$.*

Proof: We have already seen that $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ a lax map $\vartheta : \mathcal{A} \rightarrow \Sigma \mathcal{S}$. Recall that the category \mathcal{A} has as objects the sequences $\underline{u} = (u_1, \dots, u_n)$ and as morphisms the tuples $(q, \underline{f}) : \underline{u} \rightarrow \underline{v}$, where $q : \underline{n} \rightarrow \underline{m}$ is an injection and

$\underline{f} : q_*\underline{u} \rightarrow \underline{v}$ is a morphism in $(\mathcal{C}^{op})^m$. We have a functor $\mathcal{W} : \mathcal{A} \rightarrow \mathcal{C}^{op}$ defined as follows. Given an object \underline{u} of \mathcal{A} , then

$$\mathcal{W}(\underline{u}) = u_1 \otimes \cdots \otimes u_n.$$

If $(q, f) : \underline{u} \rightarrow \underline{v}$ is a morphism in \mathcal{A} , then $\underline{f} : q_*\underline{u} \rightarrow \underline{v}$ is a morphism in $(\mathcal{C}^{op})^m$, thus $f_i : u'_i \rightarrow v_i$ is a morphism in \mathcal{C}^{op} , where u'_i is either 1 or one of the u_j 's. In particular note that $\otimes_{i=1}^m u'_i = \otimes_{j=1}^n u_{\sigma(j)}$ for some permutation $\sigma \in \Sigma_n$. Define $\mathcal{W}((\sigma, \underline{f})) : \mathcal{W}(\underline{u}) \rightarrow \mathcal{W}(\underline{v})$ to be the following composite in \mathcal{C}^{op}

$$u_1 \otimes \cdots \otimes u_n \xrightarrow{\tau_{\sigma, u_1, \dots, u_n}} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)} = u'_1 \otimes \cdots \otimes u'_m \xrightarrow{f_1 \otimes \cdots \otimes f_m} v_1 \otimes \cdots \otimes v_m.$$

Where $\tau_{\sigma, u_1, \dots, u_n} : u_1 \otimes \cdots \otimes u_n \rightarrow u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$ is the coherence isomorphism given by γ in \mathcal{C}^{op} .

We claim that this functor is a strict map of permutative categories. Note that by definition it follows that $\mathcal{W}(\underline{u} \odot \underline{v}) = \mathcal{W}(\underline{u}) \otimes \mathcal{W}(\underline{v})$. On the other hand, given \underline{u} and \underline{v} object in \mathcal{A} , the symmetry isomorphism in \mathcal{A} is given by $\gamma = (\xi_{n,m}, \text{id}) : \underline{u} \odot \underline{v} \rightarrow \underline{v} \odot \underline{u}$, where $\xi_{n,m} \in \Sigma_{n,m}$ is the permutation of $(n+m)$ -letters that interchanges the first n -block with the last j -block. By definition and the coherence of the we see that $\mathcal{W}((\xi_{n,m}, \text{id})) : u_1 \otimes \cdots \otimes u_n \otimes v_1 \otimes \cdots \otimes v_m \rightarrow v_1 \otimes \cdots \otimes v_m \otimes u_1 \otimes \cdots \otimes u_n$ equals the isomorphism given by γ in \mathcal{C}^{op} . Thus we see that \mathcal{W} is a strict map. In particular, we can see \mathcal{A} and \mathcal{C}^{op} as multicategories and a lax map between them is precisely a multifunctor. Thus \mathcal{W} is an object in $\mathbf{M}(\mathcal{A}, \mathcal{C}^{op})$. If we take $\mathcal{M} = \mathcal{A}$, $\mathcal{M}' = \mathcal{C}^{op}$, $f = \mathcal{W}$ and \mathcal{E} the trivial category in Theorem 42 (which in this case agrees with (4, Corollary 12.3)) then we see that the functor

$$\mathcal{W}^* : \mathbf{M}(\mathcal{C}^{op}, \Sigma\mathcal{S}) \rightarrow \mathbf{M}(\mathcal{A}, \Sigma\mathcal{S})$$

has a left adjoint

$$\mathcal{W}_\# : \mathbf{M}(\mathcal{A}, \Sigma\mathcal{S}) \rightarrow \mathbf{M}(\mathcal{C}^{op}, \Sigma\mathcal{S})$$

such that the pair $(\mathcal{W}_\#, \mathcal{W}^*)$ forms a Quillen adjunction. Thus the image of the lax map $\vartheta : \mathcal{A} \rightarrow \Sigma\mathcal{S}$ which we call $\phi : \mathcal{C}^{op} \rightarrow \Sigma\mathcal{S}$ is a lax map. \square

7 Group completion

In this section we show that a lax map $\phi : \mathcal{C}^{op} \rightarrow \Sigma\mathcal{S}$ gives rise to lax map $\phi' : (\mathcal{C}^{op})^{-1}\mathcal{C}^{op} \rightarrow \Sigma\mathcal{S}$. (Here $(\mathcal{C}^{op})^{-1}\mathcal{C}^{op}$ denotes the Grayson-Quillen group completion of \mathcal{C}^{op}). In particular the image of the identity is a strictly commutative ring spectrum which is the ring spectrum we are looking for. We begin by recalling the definition of $(\mathcal{C}^{op})^{-1}\mathcal{C}^{op}$.

Definition 51 Let $(\mathcal{D}, \oplus, 0)$ be a small symmetric monoidal category. Then the group completion of \mathcal{D} is the category $\mathcal{D}^{-1}\mathcal{D}$, whose objects are the pairs (a, b) , where a, b are object of \mathcal{D} . The pair (a, b) is thought of as the formal difference $b - a$. If $(a, b), (c, d)$ are two objects in $\mathcal{D}^{-1}\mathcal{D}$, then a morphism in $\mathcal{D}^{-1}\mathcal{D}$, $f : (a, b) \rightarrow (c, d)$ is an equivalence class of data of the form (s, α, β) , where s is an object in \mathcal{D} and $\alpha : a \oplus s \rightarrow c$, $\beta : b \oplus s \rightarrow d$ are morphisms in \mathcal{D} . Two pairs $(s, \alpha, \beta), (s', \alpha', \beta')$ are equivalent if there exists a morphism $\gamma : s \rightarrow s'$, such that the following diagrams are commutative

$$\begin{array}{ccc} a \oplus s & \xrightarrow{\alpha} & c, \\ 1 \oplus \gamma \downarrow & \nearrow \alpha' & \\ a \oplus s' & & \end{array} \qquad \begin{array}{ccc} b \oplus s & \xrightarrow{\beta} & d \\ 1 \oplus \gamma \downarrow & \nearrow \beta' & \\ b \oplus s' & & \end{array}$$

If in \mathcal{D} every morphism is an isomorphism and for every object a of \mathcal{D} , the functor $a \oplus - : \mathcal{D} \rightarrow \mathcal{D}$ is faithful then the functor $i : \mathcal{D} \rightarrow \mathcal{D}^{-1}\mathcal{D}$ that assigns to every object a the object $(0, a)$, gives rise to a group completion on the level of classifying spaces. See (15) and (6) for a complete treatment. The following lemma follows by a straight forward computation.

Lemma 52 Let $(\mathcal{D}, \oplus, 0)$ be a small permutative category for which every morphism is an isomorphism. Then $\mathcal{D}^{-1}\mathcal{D}$ is also a permutative category and the map $i : \mathcal{D} \rightarrow \mathcal{D}^{-1}\mathcal{D}$ is a (strong) lax map.

As a particular case of the previous lemma, we have that the map $i : \mathcal{C}^{op} \rightarrow (\mathcal{C}^{op})^{-1}\mathcal{C}^{op}$ is a lax map for a permutative category \mathcal{C} . Since $\mathcal{C}^{op}, (\mathcal{C}^{op})^{-1}\mathcal{C}^{op}$ are permutative categories, we can see them as multicategories and a lax map between them is precisely a multifunctor. Thus i is an object in the category $\mathbf{M}(\mathcal{C}^{op}, (\mathcal{C}^{op})^{-1}\mathcal{C}^{op})$. If we take $\mathcal{M} = \mathcal{C}^{op}$, $\mathcal{M}' = (\mathcal{C}^{op})^{-1}\mathcal{C}^{op}$, $f = i$ and \mathcal{C}^{op} in Theorem 42 (which in this case agrees with (4, Corollary 12.3)) then we see that the functor

$$i^* : \mathbf{M}((\mathcal{C}^{op})^{-1}\mathcal{C}^{op}, \Sigma\mathcal{S}) \rightarrow \mathbf{M}(\mathcal{C}^{op}, \Sigma\mathcal{S})$$

has a left adjoint

$$i_{\#} : \mathbf{M}(\mathcal{C}^{op}, \Sigma\mathcal{S}) \rightarrow \mathbf{M}((\mathcal{C}^{op})^{-1}\mathcal{C}^{op}, \Sigma\mathcal{S})$$

such that the pair $(i_{\#}, i^*)$ forms a Quillen adjunction. We have proved the following theorem

Theorem 53 A discrete fibered symmetric bimonoidal category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ gives rise to a lax map $\phi' : (\mathcal{C}^{op})^{-1}\mathcal{C}^{op} \rightarrow \Sigma\mathcal{S}$. In particular we can associate to $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ the image of 1 under ϕ' which a strictly commutative symmetric ring spectrum.

8 Functoriality

In this section we show that the assignment of Theorem 53 is functorial.

Definition 54 *Suppose that $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ and $\Lambda' : \mathcal{D}' \rightarrow \mathcal{C}$ are two discrete fibered symmetric bimonoidal categories. A morphism from $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ to $\Lambda' : \mathcal{D}' \rightarrow \mathcal{C}$ is a functor $\Theta : \mathcal{D} \rightarrow \mathcal{D}'$ such that*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\Theta} & \mathcal{D}' \\ \Lambda \downarrow & & \swarrow \Lambda' \\ & & \mathcal{C} \end{array}$$

is a commutative diagram. In addition, we require that Θ is a lax map with respect to the operations \otimes and for each object c of \mathcal{C} , the restriction $\Theta|_{\mathcal{D}_c} : \mathcal{D}_c \rightarrow \mathcal{D}'_{\Theta(c)}$ is a lax map with respect to the operations \oplus and \oplus_c .

Definition 55 *We denote by \mathcal{FS} the category of discrete fibered symmetric bimonoidal categories and morphisms between them.*

We want to show that the assignment of Theorem 53 is functorial. To begin, suppose that $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ and $\Lambda' : \mathcal{D}' \rightarrow \mathcal{C}$ are two objects in \mathcal{FS} and $\Theta : \mathcal{D} \rightarrow \mathcal{D}'$ a morphism between them. Then after the streefication process, the functor Θ induces a strict map $\Theta^s : \mathcal{D}^s \rightarrow \mathcal{D}'^s$. The next step of the construction is to correspond functors Ψ and $\Psi' : \mathcal{A} \rightarrow \mathbb{P}$ to $\Lambda^s : \mathcal{D}^s \rightarrow \mathcal{C}^s$ and $\Lambda'^s : \mathcal{D}'^s \rightarrow \mathcal{C}^s$ respectively. Here \mathcal{A} is a certain wreath product category as defined on section 3 and \mathbb{P} is the multicategory of small permutative category. Using the map Θ^s , we can obtain a multifunctor $\Xi : \mathbb{P} \rightarrow \mathbb{P}$ by defining it to be trivial for those permutative categories not in the image of Ψ . The multifunctor Ξ is such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Psi} & \mathbb{P} \\ & \searrow \Psi' & \downarrow \Xi \\ & & \mathbb{P}. \end{array}$$

After this we apply the multifunctor K and obtain a commutative diagram of multifunctors

$$\begin{array}{ccc} E\Sigma_* & \xrightarrow{T_{\mathcal{D}}} & \Sigma \mathcal{S}^{\mathcal{A}} \\ & \searrow T_{\mathcal{D}'} & \downarrow \\ & & \Sigma \mathcal{S}^{\mathcal{A}}. \end{array}$$

The rectification performed on section 6 can be done in a functorial way and thus after passing to the completion we obtain a commutative diagram of lax

maps

$$\begin{array}{ccc} \mathcal{C}^{op}\mathcal{C} & \xrightarrow{\phi'_{\mathcal{D}}} & \Sigma\mathcal{S} \\ & \searrow \phi'_{\mathcal{D}'} & \downarrow \\ & & \Sigma\mathcal{S}. \end{array}$$

In particular, by looking at the image of the unit, we obtain a map of symmetric spectra

$$\phi'_{\mathcal{D}}(1) \rightarrow \phi'_{\mathcal{D}'}(1). \quad (10)$$

Thus we obtain the following theorem.

Theorem 56 *There is a functor*

$$\mathcal{Z} : \mathcal{FS} \rightarrow \Sigma\mathcal{CRS}$$

that for a discrete fibered symmetric bimonoidal category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ corresponds $\mathcal{Z}(\Lambda)$, the image of the unit under the lax map

$$\phi' : \mathcal{C}^{op}\mathcal{C} \rightarrow \Sigma\mathcal{S}.$$

If $\Theta : \mathcal{D} \rightarrow \mathcal{D}'$ is a morphism of discrete fibered categories, then $\mathcal{Z}(\Theta)$ is the morphism of symmetric spectra

$$\mathcal{Z}(\Lambda) \rightarrow \mathcal{Z}(\Lambda')$$

as in (10).

9 The definition

In this section we define the concept of topological fibered symmetric bimonoidal categories and show that to such each object we can correspond an E_{∞} -ring spectrum in a functorial way. This spectrum is our ultimate goal.

Definition 57 *A topological fibered category is a fibered category $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$, where \mathcal{D} and \mathcal{C} are topological categories and the functor Λ is a continuous functor that satisfies the same properties than a discrete fibered category in such a way that all the functors insight are continuous.*

In a similar way as in the discrete case we can form the category \mathcal{TFS} of topological fibered symmetric bimonoidal categories.

Remark 58 *Whenever we talk about a topological category we mean a small category \mathcal{C} whose object set is discrete and its morphism set is topological space in such a way that the structural maps are continuous.*

Suppose now that $\Lambda : \mathcal{D} \rightarrow \mathcal{C}$ is a topological fibered symmetric bimonoidal category. By applying the singular functor from topological spaces to simplicial sets on the level of objects and morphism, we can see a topological fibered symmetric bimonoidal category as a functor

$$\mathfrak{D} : \Delta^{op} \rightarrow \mathcal{FS};$$

that is, we can see a topological fibered symmetric bimonoidal category as a simplicial object in the category \mathcal{FS} . By composing \mathfrak{D} with \mathcal{Z} , we can correspond to a topological fibered symmetric bimonoidal category a functor

$$\Delta^{op} \rightarrow \Sigma\mathcal{CRS},$$

where $\Sigma\mathcal{CRS}$ is the category of strictly commutative symmetric ring spectrum. By realizing this simplicial object we obtain an E_∞ -ring spectrum. Since this construction is functorial we finally arrive to the following theorem.

Theorem 59 *There is a functor*

$$\mathfrak{J} : \mathcal{TFS} \rightarrow \mathcal{E}_\infty$$

where \mathcal{E}_∞ is the category of E_∞ -ring spectra, that for a topological fibered symmetric bimonoidal category corresponds the realization of the simplicial object in the category $\Sigma\mathcal{CRS}$ as mentioned above.

References

- [1] W. G. Dwyer and J. Spalinski. Homotopy theories and model categories. Handbook of algebraic topology, North-Holland, Amsterdam, 1995. 73-126.
- [2] A. Elmendorf, I. Kriz, M. A. Mandell and P. May. Rings, modules, and algebras in stable homotopy theory. Amer. Math. Soc. Mathematical Surveys and Monographs Vol 47. 1997.
- [3] A. Elmendorf and M. A. Mandell. Multiplicative structure in infinite loop space theory. Arxiv preprint.
- [4] A. Elmendorf and M. A. Mandell. Rings, modules and algebras in the infinite loop space theory. Adv, Math. 2005 (2006) 163-228.
- [5] J. M. Gomez, P. Hu and I. Kriz. Stringy bundles and infinite loop space theory. To appear.
- [6] D. Grayson. Higher algebraic K -theory. II (after Quillen), Algebraic K-theory (Evanston, 1976). Lec. Notes in Math., vol 551, Springer-Verlag, New York, 1976. 217-240.
- [7] M. Hovey, Model Categories. Mathematical Surveys and monographs, vol. 63, AMS, Providence, RI,1998.

- [8] M. Hovey, B. Shipley and J. Smith. Symmetric spectra. J. of the Ame. Math. Soc. Vol. 13, N. 1 Pages 149-208.
- [9] P. Hu and I. Kriz. Conformal field theory and elliptic cohomology, Adv. Math. 189 (2004) 325-412.
- [10] M. Laplaza. Coherence for distributivity. Coherence in Categories, Lecture notes in Mathematics, vol. 281, Springer, Berlin, 1972, 29-65.
- [11] M. A. Mandell, J. P. May, S. Schwede and B. Shipley. Model categories of diagram spectra. Proc. London Math. Soc. (3) 82 (2001), no. 2, 441–512.
- [12] J. P. May. The Geometry of Iterated Loop Spaces. Springer Lecture Notes in Mathematics. Springer, Berlin (1972).
- [13] J. P. May. E_∞ ring spaces and E_∞ ring spectra. With contributions by Frank Quinn, Nigel Ray, and Jorgen Tornehave. Lecture Notes in Mathematics, Vol. 577. Springer-Verlag, Berlin-New York, 1977
- [14] S. Schwede and B. E. Shipley. Algebras and modules in monoidal model categories. Proc. London Math. Soc. 2000 80(2):491-511.
- [15] R. W. Thomason. Beware of the phony Multiplication on Quillen's $\mathcal{A}^{-1}\mathcal{A}$. Proc. Amer. Math. Soc., 80(4):569-573,1980.
- [16] A. Vistoli. Notes on Grothendieck topologies, fibered categories and descent theory. Arxiv preprint.