Consider a linear transformation $T$ from $\mathbb{R}^n \rightarrow \mathbb{R}^p$ and some linearly independent vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in $\mathbb{R}^n$. Are the vectors $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_m)$ necessarily linearly independent?

A counterexample suffices: suppose $T(\vec{x}) = \vec{0}$ for all $\vec{x}$. Then $T(\vec{v}_1) = T(\vec{v}_2) = \ldots = T(\vec{v}_m) = \vec{0}$ and so they are not linearly independent.

Another way to think about it:

Since $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly independent, the only solution to $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_m \vec{v}_m = \vec{0}$ is $c_1 = c_2 = \ldots = c_m = 0$.

By the linearity of $T$, we can also write

$$k_1 T(\vec{v}_1) + k_2 T(\vec{v}_2) + \ldots + k_m T(\vec{v}_m) = \vec{0}$$

$$T(k_1 \vec{v}_1 + k_2 \vec{v}_2 + \ldots + k_m \vec{v}_m) = \vec{0}$$

In the case where $c_1 = k_1, c_2 = k_2, \ldots, c_m = k_m$, this is the same as $T(\vec{0}) = \vec{0}$, and so

$T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_m)$ are linearly independent.

If this is not the case, then we cannot draw a similar conclusion, and so $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_m)$ are not necessarily linearly independent.
#28 For which values of the constant $k$ do the vectors below form a basis of $\mathbb{R}^4$?

\[
\begin{bmatrix}
1 \\
0 \\
2 \\
2
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
3 \\
3
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
0 \\
4
\end{bmatrix}, \begin{bmatrix}
2 \\
3 \\
4 \\
k
\end{bmatrix}
\]

Vectors form a basis if they span the entire space and they are linearly independent. There are several ways to check:

- if the matrix \[
\begin{bmatrix}
1 & 0 & 0 & 27 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 4 \\
2 & 3 & 4 & k
\end{bmatrix}
\]
is invertible.

RREF: \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & k-29
\end{bmatrix}
\implies \text{this is invertible if } k \neq 29
\]

- by inspection:

\[
2 \begin{bmatrix}
1 \\
0 \\
2 \\
2
\end{bmatrix} + 3 \begin{bmatrix}
0 \\
0 \\
0 \\
3
\end{bmatrix} + 4 \begin{bmatrix}
0 \\
0 \\
0 \\
4
\end{bmatrix} = \begin{bmatrix}
2 \\
3 \\
2 \\
2
\end{bmatrix} \implies \begin{bmatrix}
2 \\
3 \\
4 \\
k
\end{bmatrix} \text{ is not linearly independent.}
\]

\[
\begin{bmatrix}
2 \\
3 \\
4 \\
29
\end{bmatrix} = \begin{bmatrix}
2 \\
3 \\
4 \\
k
\end{bmatrix} \implies \text{linearly independent if } k \neq 29
\]
#33 A subspace \( V \) of \( \mathbb{R}^n \) is called a hyperplane if \( V \) is defined by a homogeneous linear equation \( c_1x_1 + c_2x_2 + \ldots + c_nx_n = 0 \) where at least one of the coefficients \( c_i \neq 0 \). What is the dimension of a hyperplane in \( \mathbb{R}^n \) in \( \mathbb{R}^3 \) in \( \mathbb{R}^2 \)?

Since at least one \( c_i \neq 0 \), the \( x_1, x_2, \ldots, x_n \) are not linearly independent. Therefore we can remove one \( x_i \) to get a basis of the hyperplane. So, our basis has \( n-1 \) elements, and the dimension of a hyperplane in \( \mathbb{R}^n \) is \( n-1 \). In \( \mathbb{R}^3 \), this is \( 3-1 = 2 \Rightarrow \) a plane. In \( \mathbb{R}^2 \), this is \( 2-1 = 1 \Rightarrow \) a line.

3.4

#38. Find a basis \( B \) such that the \( B \) matrix of the following transformation is diagonal:

- reflection about the line in \( \mathbb{R}^2 \) spanned by \( \left[ \begin{array}{c} 2 \\ 3 \end{array} \right] \)

Recall that the reflection of a vector \( \vec{x} = \vec{x}^\perp + \vec{x}^\parallel \) is \( \text{Ref}_L(\vec{x}) = \vec{x}^\parallel - \vec{x}^\perp \) where \( \vec{x}^\parallel \) is the component parallel to the line and \( \vec{x}^\perp \) is the component perpendicular to the line. So to get a diagonal matrix (here, \( B = \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] \)) we can pick our basis to be a parallel and perpendicular vector to the line: \( \vec{v}_1 = \left[ \begin{array}{c} 2 \\ 3 \end{array} \right] \) \( \begin{bmatrix} \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow 2a + 3b = 0 \), pick \( \begin{bmatrix} a \\ b \end{bmatrix} = \left[ \begin{array}{c} -3 \\ 2 \end{array} \right] \)

\( \vec{v}_2 = \left[ \begin{array}{c} -3 \\ 2 \end{array} \right] \)

\( B = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\} \)
Consider a $3 \times 3$ matrix $A$ and a vector $\vec{v}$ in $\mathbb{R}^3$ such that $A^3 \vec{v} = \vec{0}$ but $A^2 \vec{v} \neq \vec{0}$.

(a) Show that the vectors $A^2 \vec{v}, A \vec{v}, \vec{v}$ form a basis of $\mathbb{R}^3$.

It suffices to show linear independence, i.e., the only solution to $c_1 A^2 \vec{v} + c_2 A \vec{v} + c_3 \vec{v} = \vec{0}$ is $c_1 = c_2 = c_3 = 0$.

\[ A^2 (c_1 A^2 \vec{v} + c_2 A \vec{v} + c_3 \vec{v} = \vec{0}) \]
\[ c_1 \underbrace{A^2 \vec{v}}_{\vec{0}} + c_2 \underbrace{A \vec{v}}_{\vec{0}} + c_3 \underbrace{\vec{v}}_{\vec{0}} = \vec{0} \]
\[ c_3 A^2 \vec{v} = \vec{0}, \ A^2 \vec{v} \neq \vec{0} \Rightarrow c_3 = 0. \]

\[ A (c_1 A^2 \vec{v} + c_2 A \vec{v} = \vec{0}) \]
\[ c_1 A^2 \vec{v} + c_2 A \vec{v} = \vec{0} \]
\[ c_2 A^2 \vec{v} = \vec{0}, \ A^2 \vec{v} \neq \vec{0} \Rightarrow c_2 = 0 \]

\[ c_1 A^2 \vec{v} = \vec{0}, \ A^2 \vec{v} \neq \vec{0} \Rightarrow c_1 = 0 \]

Therefore these three vectors form a basis of $\mathbb{R}^3$.

(b) Find the matrix of the transformation $T(\vec{x}) = A \vec{x}$ w.r.t. the basis $A^2 \vec{v}, A \vec{v}, \vec{v}$.

We use the following construction of $B$-matrices:

\[ B = \begin{bmatrix} [T(A^2 \vec{v})]_B & [T(A \vec{v})]_B & [T(\vec{v})]_B \end{bmatrix} \]

\[ = \begin{bmatrix} [A^2 \vec{v}]_B & [A \vec{v}]_B & [\vec{v}]_B \end{bmatrix} \]

\[ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]