This exam contains 15 pages (including this cover page) and 7 problems. Check to see if any pages are missing.

Statement of Ethics

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature ___________________________ Date ___________________________

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided.

- **No partial credit for any 1 point problem**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit.

- If you need more space, use the back of the pages; clearly indicate when you have done this.

- **Any answer or solution on scratch paper will receive no credit**.

Do not write in the table to the right.

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1 Fuentes T 1:30 - 2:20 Bloomberg 278
2 Quinan T 3:00 - 3:50 Hodson 301
3 Quinan T 4:30 - 5:20 Hodson 216
4 Luo Th 1:30 - 2:20 Bloomberg 278
5 Luo Th 3:00 - 3:50 Maryland 309

Your section number: ___________________________

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1. Let \( n \geq 2 \) be a positive integer, \( P_n \) the set of polynomials of degree less than or equal to \( n \). Prove or disprove if the following subset of \( P_n \) is a linear space. If the answer is yes, find a basis of the subspace.

(a) (1 point) \( \{a_1 t + \cdots + a_n t^n \mid a_1, \ldots, a_n \in \mathbb{R}\} \). \( \checkmark \)

No, because \( t \) and \( -t \in V \) but \( t - t = 0 \) is not!

(b) (1 point) \( \{a_0 + a_1 t + \cdots + a_n t^n \mid a_0 + \cdots + a_n = 1, a_0, \ldots, a_n \in \mathbb{R}\} \). \( \checkmark \)

No, \( t \in V \) but \( 2t \) is not.
(c) (1 point) \( \{a_0 + a_1 t + \cdots + a_n t^n \mid a_1, \ldots, a_n = 0, a_0 \in \mathbb{R} \} \).

Yes, this is \( \mathbb{R} \).

(d) (1 point) \( \{a_0 + a_1 t + \cdots + a_n t^n \mid a_0, \ldots, a_n \geq 0 \} \). \( \checkmark \)

Not a subspace.

\( t \in V \) but \( -t \notin V \).
2. Consider the following system of linear equations,

\[
\begin{align*}
2x + 3y &= 8 \\
4y + 5z &= 3 \\
6x + 7z &= -1
\end{align*}
\]

(a) (2 points) Use Gauss-Jordan elimination to solve the system of linear equations

\[
\begin{bmatrix}
2 & 3 & 0 & \mid & 8 \\
0 & 4 & 5 & \mid & 3 \\
6 & 0 & 7 & \mid & -1
\end{bmatrix} \sim \begin{bmatrix}
2 & 3 & 0 & \mid & 8 \\
0 & 4 & 5 & \mid & 3 \\
0 & -9 & 7 & \mid & -25
\end{bmatrix} \sim \begin{bmatrix}
2 & 3 & 0 & \mid & 8 \\
0 & 4 & 5 & \mid & 3 \\
0 & -1 & 17 & \mid & -19
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
2 & 0 & 5 & \mid & -49 \\
0 & 4 & 73 & \mid & -73 \\
0 & -1 & 17 & \mid & -19
\end{bmatrix} \sim \begin{bmatrix}
2 & 0 & 5 & \mid & -49 \\
0 & 0 & -1 & \mid & -1 \\
0 & -1 & 17 & \mid & 19
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
2 & 0 & 0 & \mid & 2 \\
0 & 0 & 1 & \mid & -1 \\
0 & -1 & 0 & \mid & 2
\end{bmatrix} \quad \begin{align*}
x &= 1 \\
y &= -1 \\
z &= -2
\end{align*}
\]
(b) (2 points) Use Carmer's rule to solve the system of linear equations.

Let \( A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix} \) and \( A_i \) be matrix with \( i^{th} \) column replaced by \( \begin{bmatrix} 8 \\ 3 \\ -1 \end{bmatrix} \). Then \( x_i = \frac{\det(A_i)}{\det(A)} \)

\[
\det(A) = 146 \\
\det(A_1) = 146 \\
\det(A_2) = 292 \\
\det(A_3) = -146
\]

So \( x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \) is a soln.
3. (4 points) Find the quadratic polynomial of the form \( f(x) = ax^2 + bx + c \) which fits the data \((0,0), (1,1), (-1,1), (0,1)\) best.

We need to compute \( (X^T X)^{-1} X^T y \)

where 
\[
X = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 0 & 0
\end{bmatrix} \\
y = \begin{bmatrix}
0 \\
1 \\
1 \\
1
\end{bmatrix}
\]

\[
X^T X = \begin{bmatrix}
4 & 0 & 2 \\
0 & 2 & 0 \\
2 & 0 & 2
\end{bmatrix} \quad \Rightarrow \quad (X^T X)^{-1} = \begin{bmatrix}
\frac{1}{2} & 0 & -\frac{1}{2} \\
0 & \frac{1}{2} & 0 \\
-\frac{1}{2} & 0 & 1
\end{bmatrix}
\]

\[
(X^T X)^{-1} X^T = \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} \\
0 & 0.5 & 0.5 - 0.5
\end{bmatrix}
\]

\[
(X^T X)^{-1} X^T y = \begin{bmatrix}
\frac{1}{2} \\
0 \\
\frac{1}{2}
\end{bmatrix} = \begin{bmatrix}
c \\
b \\
a
\end{bmatrix}
\]

So 
\[ f(x) = \frac{1}{2} + \frac{1}{2} x^2 \]
4. (a) (3 points) If $A$ is a $3 \times 3$ matrix, is it true that there is a line $L$ passing through the origin such that if $v \in L$ then $Av \in L$.

yes! A must have an eigenvalue $\lambda$

because $\det(A-\lambda I_3)$ is a degree 3 poly

so it has a root. The line spanned by the corresponding eigenvector has this property.
(b) (3 points) Let $A$ be a square matrix, if $A$ is diagonalizable, show that $A^2$ is diagonalizable.

Let $A = U^{-1} \Sigma U$

then $A^2 = U^{-1} \Sigma U U^{-1} \Sigma U = U^{-1} \Sigma^2 U$. 

5. Let 

\[ A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix} \]

(a) (2 points) Find all the possible eigenvalues of \( A \).

\[
\begin{align*}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 1 & 3-\lambda \end{vmatrix} \\
&= (1-\lambda)((3-\lambda)^2 - 2) - 2(2(3-\lambda) - 1) + 3(4 - (3-\lambda)) \\
&= (1-\lambda)(7 + \lambda^2 - 6\lambda) - 2(6 - 2\lambda - 1) + 3(1 + \lambda) \\
&= 7 + \lambda^2 - 6\lambda - 7\lambda - \lambda^3 + 6\lambda^2 - 10 + 4\lambda + 3 + 3\lambda \\
&= -\lambda^3 + 7\lambda^2 - 6\lambda = \lambda(-\lambda^2 + 7\lambda - 6) = -\lambda(\lambda - 1)(\lambda - 6)
\end{align*}
\]

**Eigenvalues are** 0, 1, 6

(b) (2 points) Find all the possible eigenvectors of \( A \).

\( \lambda = 0 \): \( A\mathbf{v} = 0 \) if \& only if \( \mathbf{v} \) in \( \ker(A) \). \( A \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

So \( \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \) is an eigenvector.

\( \lambda = 1 \): \( A - I \sim \begin{pmatrix} 0 & 2 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

\( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) in \( \ker(A - I) \) so \( \mathbf{v}_1 \) is eigenvector.

\( \lambda = 6 \): \( A - 6I \sim \begin{pmatrix} -5 & 2 & 1 \\ 2 & -3 & 2 \\ 3 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 2 & -3 & 2 \\ 3 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 2 & -3 & 2 \\ 3 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

So \( \mathbf{v}_2 = \begin{bmatrix} -7/11 \\ -7/11 \\ 1 \end{bmatrix} \) is eigenvector.
(c) (2 points) Is $A$ diagonalizable? Explain why.

Yes! $A = S \Lambda S^{-1}$ where $S$ is matrix of eigenvectors and $\Lambda$ diagonal matrix of eigenvalues.
6. (a) (3 points) Compute the determinant of

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
\text{a} & \text{b} & \text{c} \\
\text{a}^2 & \text{b}^2 & \text{c}^2
\end{pmatrix}.
\]

Vandermonde!

\[
(bc^2 - b^2c) - a(c^2 - b^2) + a^2(c - b)
\]

\[
= bc^2 - b^2c - a(c - b)(c + b) + a^2(c - b)
\]

\[
= (c - b) \left[ bc - a(c + b) + a^2 \right]
\]

\[
= (c - b) \left[ a^2 - (c + b)a + bc \right]
\]

\[
= (c - b)(a - c)(a - b)
\]

\[\text{Quadratic Equation!}\]
(b) (3 points) Find the classical adjoint of the matrix

\[
A = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}.
\]

and use the result to find \(A^{-1}\).

\[
\text{Ad}_{j}(A)_{11} = 1 \quad \text{Ad}_{j}(A)_{21} = 0 \quad \text{Ad}_{j}(A)_{31} = -1
\]

\[
\text{Ad}_{j}(A)_{12} = 0 \quad \text{Ad}_{j}(A)_{22} = -1 \quad \text{Ad}_{j}(A)_{23} = 0
\]

\[
\text{Ad}_{j}(A)_{13} = -2 \quad \text{Ad}_{j}(A)_{23} = 0 \quad \text{Ad}_{j}(A)_{33} = 1
\]

So \( \text{Ad}_{j}(A)^{\top} = \begin{bmatrix}
1 & 0 & -1 \\
0 & -1 & 0 \\
-2 & 0 & 1
\end{bmatrix} \)

\( \det(A) = 1 + 2(-1) = -1 \)

\[
\therefore A^{-1} = \frac{1}{\det(A)} \text{Ad}_{j}(A)^{\top} = \begin{bmatrix}
-1 & 0 & 1 \\
0 & 1 & 0 \\
2 & 0 & -1
\end{bmatrix}
\]
7. (a) (5 points) Let $B_n$ be an $n \times n$ matrix with 0's along the diagonal and 1's everywhere else,

$$B_n = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}.$$ 

Compute $\det B_n$. 

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 3 & 3 & 3 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \Rightarrow 3(-1)^3$$

$$(-1)^{n-1}(-1)^{n-1}$$

$$(-1)$$
(b) (5 points) There are $n$ students in the course: AS 201 Linear Algebra. Suppose that any $n - 1$ students can be divided into 2 groups with the same total weights. Is it possible that $n$ is an even number?
Please estimate your score. You can also provide your suggestion for the course here.