1. The geometry of Euclidean space

1.1. Vectors in 2 or 3 dim space.

Definition 1.1 (Vector). Geometry: consists of a direction and a length. Represented by a line segment with a definite direction.

Algebra: represent vector in terms of its component along coordinate axis in \( \mathbb{R}^n \).

Example 1.1. Length of \( |A| \). \( |(1,2,3)| = \sqrt{14} \). If \( A = (a_1, a_2, a_3) \), then

\[
|A| = \sqrt{a_1^2 + a_2^2 + a_3^2}.
\]

Addition of vectors, \( A + B \). One can define it by geometry and algebra.

\( A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \). We can define \( \lambda A \) for \( \lambda \in \mathbb{R} \). What’s \( A - B \) ?

1.2. Inner product.

Theorem 1.2.

\[
A \cdot B := \sum a_i b_i = |A||B| \cos \theta,
\]

where \( \cdot \) is the inner product and \( \theta \) is the angle between the vectors \( A \) and \( B \).

In textbook, vector also denote by \( a \), and the inner product is denoted by \( (a, b) \).

Special case of Theorem 1.2

\[
A \cdot A = |A|^2
\]

Proof. We have,

\[
|c|^2 = c \cdot c = (a - b) \cdot (a - b) = a \cdot a + b \cdot b - 2a \cdot b = |a|^2 + |b|^2 - 2a \cdot b.
\]

By Law of cosines:

\[
|c|^2 = |a|^2 + |b|^2 - 2|a||b| \cos \theta.
\]

Thus,

\[
a \cdot b = |a||b| \cos \theta.
\]

Application of Theorem 1.2

(1) compute length and angels,
(2) detect orthogonality, and
(3) the length of $a$ along $u$.

1.3. Matrices, Discriminant and the cross product. Area and volume.

How to compute the area of a polygon?

Idea: divide it into a sum of triangles. Reduce the problem to compute
the area of an triangle.

Let $a, b$ be the vectors, then the area of triangle determined by the two
vectors is

$$\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = |a_1b_2 - a_2b_1|.$$  

Determinant in Space.

The volume of parallelepiped determined by $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$
and $c = (c_1, c_2, c_3)$ is

$$\text{vol} = |a \cdot (b \times c)| = |\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}|.$$

Here

$$b \times c = \begin{pmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$  

$\times$ is the cross product, and we have the following properties:

(1) $|b \times c|$ is the area of parallelogram which is determined by vectors $b$
and $c$, e.g. $b \times b = 0$, and

(2) the direction of $b \times c$ satisfying the right-hand rule: right hand points
$b$, fingers points $c$, and the thumb points $b \times c$. e.g. $j \times i = -k$.

Attention: $a \times b \neq b \times a$ if $a \neq b$. Actually, $a \times b = -b \times a$.

Distance: point to plane

Equation of a plane in the space:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

where $n = Ai + Bj + Ck$ is the normal vector, $i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1)$.

Distance from a point $(x_1, y_1, z_1)$ to the plane $A(x - x_0) + B(y - y_0) +
C(z - z_0) = 0$ is

$$\frac{|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|}{\sqrt{A^2 + B^2 + C^2}}.$$  

Example 1.2. The distance from $(2, 0, -1)$ to $3x - 2y + 8z + 1$ is $\frac{1}{\sqrt{11}}$.

1.4. Cylindrical and Spherical Coordinates. $(x, y) \rightarrow (r \cos \theta, r \sin \theta)$,
$r \geq 0, 0 \leq \theta < 2\pi$.

Cylindrical Coordinates: $(x, y, z) \rightarrow (r \cos \theta, r \sin \theta, z), r \geq 0, 0 \leq \theta < 2\pi$.

Let $(x, y, z) \rightarrow (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi), \rho \geq 0, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi$. 

1.5. **n-dim Vectors.** $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The length (or norm) of a vector $\mathbf{x}$:

$$||\mathbf{x}|| = \sqrt{x_1^2 + \cdots + x_n^2},$$

Cauchy inequality:

$$||\mathbf{x}|| ||\mathbf{y}|| \geq \mathbf{x} \cdot \mathbf{y},$$

where $||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

Several methods:

1. Theorem of Inner product, $\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}|| ||\mathbf{y}|| \cos \theta \leq ||\mathbf{x}|| ||\mathbf{y}||$.

2. $$\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - (\sum_{i=1}^{n} x_i y_i)^2 = \sum_{1 \leq i < j \leq n} (x_i y_j - x_j y_i)^2,$$

and

3. $$(\mathbf{x} - \lambda \mathbf{y}) \cdot (\mathbf{x} - \lambda \mathbf{y}) \geq 0.$$ 

Application: the triangle inequality,

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||.$$

2. **Differentiation**

2.1. **the Geometry of Real-Valued Functions.** Function $f : \text{domain } U \subseteq \mathbb{R}^n, \text{range } \subseteq \mathbb{R}^m$. $f$ is vector valued if $m \geq 1$, is scalar valued if $m = 1$.

**Definition 2.1.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, graph of $f$ is defined by

$$\text{graph}(f) := \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n))|(x_1, \ldots, x_n) \in U\}.$$ 

2.2. **Limits and Continuity.** $\mathbf{x}_0 \in \mathbb{R}^n$, open disk of radius $r$ is defined by

$$\mathcal{B}_r(\mathbf{x}_0) := ||\mathbf{x} - \mathbf{x}_0|| < r.$$ 

**Definition 2.2.** We say $U \subseteq \mathbb{R}^n$ is an open set in $\mathbb{R}^n$, if for any point $\mathbf{x}_0 \in U$, there exists $r > 0$, such that $\mathcal{B}_r(\mathbf{x}_0) \subseteq U$.

A neighborhood of $\mathbf{x} \in \mathbb{R}^n$ is an open set $U$ containing $\mathbf{x}$.

**Definition 2.3.** We say $V \subseteq \mathbb{R}^n$ is a closed set in $\mathbb{R}^n$, if $V^c := \mathbb{R}^n \setminus V$ is open.

**Example 2.4.** $(0, 1) \subset \mathbb{R}^1$ is open, $[0, 1] \subset \mathbb{R}^1$ is closed, $(0, 1] \subset \mathbb{R}^1$ neither open nor closed. $(0, 1) \subset \mathbb{R}^2$ is Not open.

**Definition 2.5 (Limits).** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, we say $\mathbf{y} \in \mathbb{R}^m$ is a limit of $f(\mathbf{x})$ at $\mathbf{x}_0$ if for any $\epsilon > 0$, there exists a positive real number $\delta > 0$ which depends on $\epsilon$, such that for any $0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta$, we have

$$||f(\mathbf{x}) - \mathbf{y}|| < \epsilon.$$ 

We denote it by

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{y}.$$
Remark 2.6. Caution. $0 < ||x-x_0|| < \delta$. It is possible that $\lim_{x \to x_0} f(x) \neq f(x_0)$. e.g., $f(x) = 1, x \neq 0, f(x) = 0, x = 0$.

Properties of limits.
(1) Uniqueness. $\lim_{x \to x_0} f(x) = y_1$, $\lim_{x \to x_0} f(x) = y_2$ implies $y_1 = y_2$.
(2) Scalar multi. $\alpha \in \mathbb{R}$. $\lim_{x \to x_0} \alpha f(x) = \alpha \lim_{x \to x_0} f(x)$.
(3) Addition. $\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$.
(4) Multiplication. $(m = 1)$ $\lim_{x \to x_0} f(x)g(x) = \lim_{x \to x_0} f(x) \lim_{x \to x_0} g(x)$.
(5) Division. $(m = 1)$ $\lim_{x \to x_0} \frac{f(x)}{x} = \frac{1}{\lim_{x \to x_0} f(x)}$.

Example 2.7. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) := x^2 + y^2 + 2$.

$$\lim_{(x,y) \to (0,1)} f(x,y) = 3.$$ 

Example 2.8.

$$\lim_{(x,y) \to (0,1)} e^{xy} = 1.$$ 

Example 2.9. $\lim_{(x,y) \to (0,0)} \frac{2xy}{x^2 + y^2}$ does not exist.

Definition 2.10. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$. We say that $f$ is continuous at $x_0$ if and only if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Example 2.11. $f(x) = 1, x \neq 0, f(x) = 0, x = 0$ is not continuous at 0.

Example 2.12. Any polynomial is continuous.

Theorem 2.1. $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $x \to (f_1(x), f_2(x), \ldots, f_n(x))$ is continuous if and only if $f_i(x)$ is continuous for any $i$.

Example 2.13. $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x,y) = (x^2y, \frac{y+xy^3}{1+x^2})$ is continuous.

Theorem 2.2. Let $g : V \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^l$. Suppose $g(V) \subseteq U$, if $g$ is continuous at $x_0 \in V$, and if $f$ is continuous at $y_0 = g(x_0)$, then $f(g(x))$ is continuous at $x_0$.

Intuitively, as $x$ gets close to $x_0$, $g(x)$ gets close to $g(x_0)$, and as $g(x)$ gets close to $g(x_0)$, $f(g(x))$ get close to $f(g(x_0))$.

Example 2.14. $f(x,y,z) = (x^2 + y^2 + z^2)^{30} + \sin z^3$. 
2.3. Differentiation.

**Definition 2.15** (Partial derivatives). Let $U \subseteq \mathbb{R}^n$ be an open set and suppose that $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is a real-valued function. Then the partial derivatives of $f$ with respect to the first, second, ..., $n$th variable, are the real valued functions of $n$ variables, which, at the point $x_0 = (x_1^0, \ldots, x_n^0)$ are defined by

$$\frac{\partial f}{\partial x_j}(x_1^0, \ldots, x_n^0) = \lim_{h \to 0} \frac{f(x_0 + h e_j) - f(x_0)}{h}.$$ 

In other words, $\frac{\partial f}{\partial x_j}$ is the derivative of $f$ with respect to the variable $x_j$, with the other variables held fixed.

**Example 2.16.** $f(x, y) = \sin x + y^2$. $\frac{\partial f}{\partial x} = \cos x$, $\frac{\partial f}{\partial y} = 2y$.

It is possible that some partial derivative may exist others not.

**Example 2.17.** $g(x, y) = |x| - y$. $\frac{\partial g}{\partial x}$ does not exist when $x = 0$. $\frac{\partial g}{\partial y} = -1$.

The next example shows that the existence of partial derivatives does not imply that $f$ is continuous.

**Example 2.18.** Let $f(x, y) = \frac{2xy}{x^2 + y^2}$, $(x, y) \neq (0, 0)$; $f(0, 0) = 0$. Then $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2}$ does not exist, and $\frac{\partial f}{\partial x}(0, 0) = 0 = \frac{\partial f}{\partial y}(0, 0)$.

**Example 2.19.** $f(x, y) = \frac{(y^2 - x^2)^2}{y^4 + x^2}$.

For one variable function $f(x)$, $f$ is differentiable at a point $x_0$, then

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0.$$ 

The tangent line $l$ through $(x_0, f(x_0))$ with slope $f'(x_0)$ is close to $f$ near $(x_0, f(x_0))$ even when divided by $x - x_0$.

**Definition 2.20** (Differentiable: Two variables). Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. We say $f$ is differentiable at $(x_0, y_0)$, if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(x_0, y_0)$ and if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)}{||(x,y) - (x_0,y_0)||} = 0.$$ 

**Definition 2.21** (Tangent plane). $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be differentiable at $(x_0, y_0)$, the tangent plane is defined by

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$ 

**Definition 2.22** (Differentiable). Let $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$. We say $f$ is differentiable at $x_0 \in U$, if partial derivatives of $f$ exist at $x_0$ and if

$$\lim_{x \to x_0} \frac{||f(x) - f(x_0) - Df(x_0)(x - x_0)||}{||x - x_0||} = 0.$$
Where $Df(x_0)$ is the $m \times n$ matrix with elements $\frac{\partial f_i}{\partial x_j}$ evaluated at $x_0$. $Df$ is called the Jacobian matrix of $f$. When $m = 1$, $Df(x) = (\frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_n}{\partial x_n})$ is called the gradient of $f$, and denoted by $\nabla f$ or $\text{grad} f$.

**Remark 2.23.** $f$ is differentiable at $x_0 \in U$ if and only if $f_i$ is differentiable at $x_0 \in U$ for any $i$.

**Theorem 2.24.** Let $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$.

1. Suppose the partial derivatives $\frac{\partial f_i}{\partial x_j}$ all exist and are continuous in a neighborhood of a point $x_0 \in U$ ($C^1$ function). Then $f$ is differentiable at $x_0$.
2. If $f$ is differentiable at $x_0$, then $f$ is continuous at $x_0$.

**Example 2.25.**

- The Jacobian matrix of $f(x, y) = (e^{x+y} + y, y^2 x)$. $f(x, y) = 1, x = 0$ or $y = 0$, $f(x, y) = 0$ otherwise; $f(x, y)$ is not continuous at $(0, 0)$.
- $f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) = e^{xy} + \sin xy$. $\nabla f(x, y) = (e^{xy} + \cos xy)(yi + xj)$.

### 2.4. Paths and curves.

**Definition 2.26.** A path in $\mathbb{R}^n$ is a continuous map $f : [a, b] \rightarrow \mathbb{R}^n$. The image $f([a, b])$ is a curve.

**Example 2.27.**

- The unit circle $x^2 + y^2 = 1$ is a curve.
- $f(t) = (t, t^2)$, the graph of $f(x) = x^2$.

**Definition 2.28** (Velocity Vector). If $f(t)$ is a path and it is differentiable, we say $f(t)$ is a differentiable path. The velocity of $f$ at time $t$ is defined by

$$f'(t) = \lim_{h \rightarrow 0} \frac{f(t + h) - f(t)}{h}.$$  

$f'(t)$ is a vector tangent to the path $f(t)$ at time $t$. The speed of the path $f(t)$ is $||f'(t)||$, the length of the velocity vector. If $f(t) = (x_1(t), \ldots, x_n(t))$, $f'(t) = (x'_1(t), \ldots, x'_n(t)).$

**Example 2.29.** Compute the tangent vector to the path $f(t) = (t, t^2, e^t)$ at $t = 0, (1, 0, 1)$.

**Definition 2.30.** If $f(t)$ is a path, and if $f'(t_0) \neq 0$, the equation of its tangent line at the point $f(t_0)$ is

$$I(t) = f(t_0) + (t - t_0)f'(t_0).$$

**Example 2.31.** Find the tangent line to $f(t) = (t \sin t, 4t)$ at $t = 0, (0, 4t)$

### 2.5. Chain Rule.

For one variable, $\frac{df(g(t))}{dt} = f'(g(t))g'(t)$. For two variables, $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Suppose $g(x, y) = (u(x, y), v(x, y))$, $g$ is differentiable at $(x_0, y_0)$, $f$ is differentiable at $g(x_0, y_0)$. Let $h := f \circ g(x, y) = f(u(x, y), v(x, y))$, then

$$\left[\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right] = \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right] \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right] \left[\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right]$$
Example 2.32. \( f(x, y) = xy, \ g(t) = (e^t, \cos t). \ h = f \circ g(t) \).
\[ h(t) = f(e^t, \cos t) = e^t \cos t, \]
\[ h'(t) = e^t \cos t - e^t \sin t. \]

Example 2.33. \( g(x, y) = (x^2 + 1, y^2), \ f(u, v) = (u + v, u, v^2) \).
\[ Df(u, v)Dg(x, y) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{bmatrix} \begin{bmatrix} 2x \\ 0 \\ 2y \end{bmatrix} \]

Theorem 2.3 (Chain Rule). Let \( U \subset \mathbb{R}^n \) and \( V \subset \mathbb{R}^m \) be open sets.
\( g : U \subset \mathbb{R}^n \to \mathbb{R}^m, \ f : \subset \mathbb{R}^m \to \mathbb{R}^p \) be given functions such that \( g(U) \subset V \).
Suppose \( g \) is differentiable at \( x_0 \), and \( f \) is differentiable at \( y_0 = g(x_0) \), then
\( f \circ g \) is differentiable at \( x_0 \), and
\[ D(f \circ g)(x_0) = Df(y_0)Dg(x_0). \]
The right hand side is the matrix product.

Example 2.34. \( f(x, y, z) = u^2 + v^2 - w, \ u = x^2y, \ v = y^2, \ w = e^{-xz}. \)
\( h(x, y, z) = x^4y^2 + y^4 - e^{-xz}. \)
\( \frac{\partial h}{\partial x} = 2u \cdot 2xy + 2v \cdot 0 + (-1)(-ze^{-xz}) = 4x^3y^2 + ze^{-xz}. \)

2.6. Gradients and Directional Derivatives.

Definition 2.35 (the Gradient). If \( f : U \subseteq \mathbb{R}^3 \to \mathbb{R} \) is differentiable, the
gradient of \( f \) at \((x, y, z)\) is the vector in space given by
\[ \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}). \]

Definition 2.36 (Directional Derivatives). If \( f : \mathbb{R}^3 \to \mathbb{R} \), the directional
derivative of \( f \) at \( x \) along the vector \( v \) is given by \( \frac{d}{dt}(f(x + tv))|_{t=0} \) if this exists.

Example 2.37. \( f(x, y) = x^2 - y, \ v = (1, 2). \) \( \frac{d}{dt}(f(x + tv))|_{t=0} = 2x - 2. \)
Suppose \( f \) is differentiable, then all directional derivatives exist. By Chain rule, we can show
\[ \frac{d}{dt}(f(x + tv))|_{t=0} = D_v f(x) = \nabla f(x) \cdot v. \]

Theorem 2.38 (The gradient is normal to level surfaces). Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be a \( C^1 \) map and let \((x_0, y_0, z_0)\) lie on the level surface \( S \) defined by \( f(x, y, z) = k \), for \( k \) a constant. Then \( \nabla f(x_0, y_0, z_0) \) is normal to the level surface in the following sense: if \( v \) is the tangent vector at \( t = 0 \) of a path \( c(t) \) in \( S \) with \( c(0) = (x_0, y_0, z_0) \), then \( \nabla f(x_0, y_0, z_0) \cdot v = 0 \). The tangent plane of \( S \) at a point \((x_0, y_0, z_0)\) of \( S \) is defined by equation \( \nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0 \).

Proof. By the chain rule, \( 0 = \frac{d}{dt}(f(c(t)))|_{t=0} = \nabla f(c(0)) \cdot v. \)

Example 2.39. The equation of the plane tangent to the surface defined by \( 3xy + z^2 = 4 \) at \((1, 1, 1)\) is \( 3x + 3y + 2z = 8 \).
3. Higher-order Derivative: Maxima and Minima

3.1. Iterated Partial Derivatives.

Definition 3.1. A function \( f : U \to \mathbb{R} \) is of class \( C^1 \) if \( \nabla(f) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \) exists and is continuous at each point of \( U \). If each partial derivative of \( \frac{\partial f}{\partial x_i} \) exists and is continuous at each point of \( U \), then we say \( f \) is of class \( C^2 \).

Definition 3.2. \( \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y} \) are called iterated partial derivatives, \( \frac{\partial^2 f}{\partial y \partial x} \) are called mixed partial derivatives.

Example 3.3. \( f(x, y) = e^{xy} \). \( f(x, y) = \sin x \sin^2 y \).

Theorem 3.4 (Equality of Mixed Partial). If \( f(x, y) \) is of class \( C^2 \), then
\[
\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.
\]
Proof.
\[
f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) - f(x_0, y_0 + \Delta y) + f(x_0, y_0)
= \frac{\partial^2 f}{\partial y \partial x}(x', y')\Delta x \Delta y.
\]
\[\Box\]

Example 3.5. \( f(x, y) = xe^y + yx^2 \).

Theorem 3.6 (Equality of Mixed Partial). If \( f(x, y, z) \) is of class \( C^3 \), then
\[
\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial^3 f}{\partial z \partial y \partial x}.
\]

3.2. Taylor’s Theorem.

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + h_k(x)(x - x_0)^k,
\]
where \( \lim_{x \to x_0} h_k(x) = 0 \).

Example 3.7. \( f(x) = e^{2x} \) at \( x = 0 \), \( f(x) = 1 + 2x + 2x^2 + h_2(x)x^2 \).

Generalize to multi-variable.

Theorem 3.8 (First-order Taylor Formula). Let \( f : U \subset \mathbb{R}^n \to \mathbb{R} \) be differentiable at \( x_0 \in U \). Then
\[
f(x_0 + h) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + R_1(x_0, h),
\]
where \( \frac{R_1(x_0, h)}{\|h\|} \to 0 \) as \( h \to 0 \) in \( \mathbb{R}^n \).
Theorem 3.9 (Second-order Taylor Formula). Let \( f : U \subset \mathbb{R}^n \to \mathbb{R} \) is of class \( C^2 \). Then

\[
f(x_0 + h) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + \frac{1}{2} \sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) + R_2(x_0, h),
\]

where \( \frac{R_2(x_0, h)}{||h||^2} \to 0 \) as \( h \to 0 \) in \( \mathbb{R}^n \).

Example 3.10. \( f(x, y) = e^{2x+y} \), find second-order expansion at \((0,0)\). 
\( f(x, y) = 1 + 2x + y + 4x^2 + 4xy + y^2 + R_2 \).

Example 3.11. \( f(x, y) = \sin xy \) at \((x_0, y_0) = (1, \frac{\pi}{2})\). 
\[ 1 - \frac{x^2}{8} \left( x - 1 \right)^2 - \frac{\pi}{2} \left( x - 1 \right) \left( y - \frac{\pi}{2} \right) - \frac{1}{2} \left( y - \frac{\pi}{2} \right)^2. \]

3.3. Maxima and Minima for functions of \( n \)-variables.

Definition 3.12. If \( f : U \subset \mathbb{R}^n \to \mathbb{R} \) is a given scalar function (\( U \) is an open set), a point \( x_0 \in U \) is called a local minimum of \( f \) if there is a neighborhood \( V \) of \( x_0 \) such that for all \( x \in V \), \( f(x) \geq f(x_0) \). A point \( x_0 \) is a critical point of \( f \) if either \( f \) is not differentiable at \( x_0 \), or it is, \( Df(x_0) = 0 \).

Theorem 3.13. Every extremum is a critical point.

Proof. \( g(t) = f(x_0 + th) \). \( g'(0) = 0 = Df(x_0) \cdot h. \)

Definition 3.14. A critical point that is not a local extremum is called a saddle point.

Example 3.15. \( f(x) = x^3 \) at \( x = 0 \). \( f(x, y) = x^2 - y^2 \) at \((x, y) = (0,0)\). 
\( f(x, y) = x^2 + y^2 \) at \((x, y) = (0,0)\).

Definition 3.16. Suppose \( f \) is \( C^2 \). The Hessian of \( f \) at \( x_0 \) is the quadratic function defined by

\[
Hf(x_0)(h) = \frac{1}{2} \sum_{1 \leq i,j \leq n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)h_i h_j.
\]

A quadratic function is called positive-definite if \( g(h) \geq 0 \) for all \( h \in \mathbb{R}^n \) and \( g(h) = 0 \) only for \( h = 0 \).

If \( Df(x_0) = 0 \), then by the Taylor formula,

\[
f(x_0 + h) = f(x_0) + Hf(x_0)(h) + R_2(x_0, h).
\]

Theorem 3.17. If \( f : U \subset \mathbb{R}^n \to \mathbb{R} \) is \( C^2 \), \( x_0 \) is a critical point of \( f \), and the Hessian \( Hf(x_0)(h) \) is positive-definite, then \( x_0 \) is a relative minimum of \( f \). (Maximum is similar)

Proof. \( Hf(x_0)(h) \geq M||h||^2 \) for some \( M > 0 \). 
\[ 0 < Hf(x_0)(h) + R_2(x_0, h) = f(x_0 + h) - f(x_0) \]

In particular, we have the following.
**Theorem 3.18.** $f(x,y)$ is $C^2$ on an open set $U \subset \mathbb{R}^2$. Suppose that 
\[
\frac{\partial f}{\partial x}(x_0, y_0) = 0 = \frac{\partial f}{\partial y}(x_0, y_0). \quad \frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0. \quad \text{Then } (x_0, y_0) \text{ is a local minimum of } f, \text{ if } D = \frac{\partial^2 f}{\partial x^2}(x_0, y_0) \frac{\partial^2 f}{\partial y^2}(x_0, y_0) - \left( \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \right)^2 > 0. \quad \text{If } D < 0, \text{ then } (x_0, y_0) \text{ is a saddle point.}
\]

**Example 3.19.** $f(x, y) = x^2 + y^2$. $f(x, y) = x^5 y + xy^5 + xy$, saddle point. 
$f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$, $(1, 1)$ is local min. 

Global extreme.

**Theorem 3.20** (Global existence Theorem for Maxima and Minima). A set $D$ is said to be bounded if there is a number $M > 0$, such that $||x|| < M$ for all $x \in D$. Let $D$ be a closed and bounded set in $\mathbb{R}^n$ and let $f : D \to \mathbb{R}$ be continuous. Then $f$ attains its maximum and minimum values at some points of $D$.

**Example 3.21.** Find the Max and min value of the function $f(x, y) = x^2 + y^2 - x - y + 1$ in the Disc $x^2 + y^2 \leq 1$. $-\sqrt{2} \leq -x - y \leq \sqrt{2}$. 
min $= \frac{1}{2}, \max = 2 + \sqrt{2}$.

### 3.4. Constrained Extrema and Lagrange Multipliers.

**Theorem 3.22** (Lagrange Multipliers). Suppose that $f : U \subset \mathbb{R}^n \to \mathbb{R}$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}$ are given $C^1$ real-valued functions. Let $x_0 \in U$ and 
$g(x_0) = c$, and let $S$ be the level set for $g$ with value $c$. Assume $\nabla g(x_0) \neq 0_n$. 

If $f|_S$, which denotes “$f$ restricted to $S$” has a local maximum or minimum on $S$ at $x_0$, then there is a real number $\lambda$ such that $\nabla f(x_0) = \lambda \nabla g(x_0)$.

**Proof.** Some geometry from the proof. For any path $c(t)$ in $S$, $\frac{\partial}{\partial t} g(c(t)) = \frac{dc}{dt} = 0 = \nabla g(x_0) \cdot c'(0)$, $c'(0)$ is orthogonal to $\nabla g(x_0)$.

If $f|_S$ has a maximum at $x_0$, then $f(c(t))$ has a maximum at $t = 0$. 
$0 = \frac{\partial}{\partial t} g(c(t)) = \nabla f(x_0) \cdot c'(0) \quad \nabla f(x_0)$ and $\nabla g(x_0)$ are parallel. \hspace{1cm} \Box

**Example 3.23.** $f(x, y) = x^2 - y^2$, $S : x^2 + y^2 = 1$. Find the extrema of $f|_S$. $(0, 1), (0, -1), (1, 0), (-1, 0)$.

several constraints.

**Example 3.24.** $f(x, y, z) = x+y+z$, $x^2 + y^2 = 2, x+z = 1$. $(0, \sqrt{2}, 1), (0, -\sqrt{2}, 1)$.

Strategy for finding max and min on bounded regions with boundary. Let $f$ be a differentiable function on a closed and bounded region $D = U \cup \partial U$.

1. Locate all critical point of $f$ in $U$.
2. Use Lagrange multiplier to locate all the critical points of $f|_{\partial U}$.
3. Compute the values of $f$ at all these critical points.
4. select the largest and the smallest.

**Example 3.25.** Find the absolute maximum of $f(x, y) = xy$ on the unit disc $D$, where $D$ is the set of points $(x, y)$ with $x^2 + y^2 \leq 1$.

Optional: Second-Derivative test for constrained extrema.
4. Vector-Valued Functions

Recall a path in \( \mathbb{R}^n \) is a map of \( c \) of \( \mathbb{R} \) or an interval in \( \mathbb{R}^n \). If a path is differentiable, \( c'(t) = (c'_1(t), \ldots, c'_n(t)) \).

**Example 4.1.** \( b(t) = (t, 3t^2), c(t) = (4e^t, \sin(-t)). \) \( \frac{d}{dt}(b(t) \cdot c(t)) = (4e^t + te^{4t}, 6t \sin + 3t^2 \cos(t)) = b'(t) \cdot c(t) + b(t) \cdot c'(t). \)

**Theorem 4.2.** \( \frac{d}{dt}(b(t) \cdot c(t)) = b'(t) \cdot c(t) + b(t) \cdot c'(t). \) \( \frac{d}{dt}(b(t) \times c(t)) = b'(t) \times c(t) + b(t) \times c'(t). \)

**Example 4.3.** If \( c(t) \) is a vector function such that \( ||c(t)|| \) is constant, then \( c'(t) \) is perpendicular to \( c(t) \) for all \( t \).

**Theorem 4.4.** If \( c(t) \) is \( C^1 \). The length of the path \( c(t) = (x(t), y(t), z(t)) \) for \( t_0 \leq t \leq t_1 \) is 

\[
c(t) = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.
\]

**Example 4.5.** The arc length of the path \( c(t) = (r \cos t, r \sin t) \) for \( t \) lying in the interval \([0, 2\pi]\) is \( 2\pi r \). The graph of a function of one variable \( y = f(x) \) for \( x \in [a, b] \) is \( \int_a^b \sqrt{1 + f'(x)^2} dx \). \( c(t) = (x(t), y(t)) = (|t|, |t - \frac{1}{2}|), -1 \leq t \leq 1. \)

**Definition 4.6 (Vector fields).** A vector fields in \( \mathbb{R}^n \) is a map \( F : A \subset \mathbb{R}^n \to \mathbb{R}^n \) that assigns to each point \( x \) in its domain \( A \) a vector \( F(x) \). If \( n = 2, F \) as called a vector field in the plane, and \( n = 3, F \) is a vector field in a space.

**Example 4.7.** \( V(x, y) = -yi + xj \). Gradient vector Fields, \( \nabla f(x, y, z) \). Show that \( V(x, y) = yi - xj \) is not a gradient vector field.

**Definition 4.8 (divergence, curl).** del operator \( \nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \).

\( F = (F_1, F_2, F_3), \) \( \text{div} F = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}, \text{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}, \) Laplace operator \( \nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \)

**Example 4.9.** \( F = x^2yi + zj + xyk \). \( \text{div} F = 3xy \). \( F = xi + xyj + k \). \( \nabla \times F = yk. \)

**Theorem 4.10.** \( f \) is \( C^2 \). \( \nabla \times (\nabla f) = 0 \). \( \text{div(curl} F) = \nabla \cdot (\nabla \times F) = 0 \)

**Example 4.11.** \( V(x, y) = yi - xj \) is not a gradient vector field since \( \text{curl} V = -2k \). \( V = xi + yj + zk \) cannot be the curl of some vector field \( F \) since \( \text{div} V = 3 \neq 0 \). \( \nabla^2 (\frac{1}{\sqrt{x^2+y^2+z^2}}) = 0 \). If \( \nabla^2 f = 0, f \) is harmonic.

**Theorem 4.12.** If \( f \) is a harmonic function defined on all of \( \mathbb{R}^n \) which is bounded above or bounded below, then \( f \) is constant. If \( V \) is a nonempty closed and bounded subset of \( U \), then \( f \) restricted to \( V \) attains its maximum and minimum on the boundary of \( V \). If \( f \) is harmonic, then \( F(z) = F(x + iy) = f_x - if_y \) is differentiable.
5. Double and triple integrals

Let \( f \) be a bounded function, i.e., there is a number \( M > 0 \) such that 
\[-M < f(x, y) < M \] for all \((x, y)\) in the domain of \( f \). Let \( a = x_1 \leq x_2 \leq \cdots \leq x_n \leq b = x_{n+1} \), and \( x_{i+1} - x_i = \frac{b-a}{n} \), \( c_i \in [x_i, x_{i+1}] \), \( S_n = \sum_{i=1}^{n} f(c_i) \frac{b-a}{n} = \sum_{i=1}^{n} f(c_i) \Delta A \). \( S_n \) is called Riemann sum of \( f \).

**Definition 5.1.** If the sequence \( S_n \) converge to a limit \( S \) as \( n \to \infty \) and if the limit \( S \) is the same for any choice of point \( c_i \) in the closed interval, then we say \( f \) is integral over \([a, b]\) and we write \( \int_{a}^{b} f(x) \, dx \).

The area under the graph of a nonnegative continuous function \( f \) from \( x = a \) to \( x = b \) is \( \int_{a}^{b} f(x) \, dx \).

Nonnegative is important, e.g. \( f(x) = 1 - x \).

Let \( R \) be a rectangle \( R = [a, b] \times [c, d] \subset \mathbb{R}^2 \), \( a = x_1 \leq x_2 \leq \cdots \leq x_n \leq b = x_{n+1} \), \( c = y_1 \leq y_2 \leq \cdots \leq y_n \leq d = y_{n+1} \), \( c_{i,j} \in [x_i, x_{i+1}] \times [y_j, y_{j+1}] \).

\[ S_n = \sum_{i=1}^{n} S_{i,j} = \sum_{i=1}^{n} f(c_{i,j}) \Delta A \].

**Definition 5.2.** If the sequence \( S_n \) converge to a limit \( S \) as \( n \to \infty \) and if the limit \( S \) is the same for any choice of point \( c_{i,j} \) in the rectangles, then we say \( f \) is integral over \( R \) and we write \( \int \int_{R} f(x, y) \, dxdy, \int \int_{R} f(x, y) \, dA \) or \( \int_{R} f(x, y) \, dxdy \) for the limit \( S \).

**Definition 5.3.** The volume of the region above \( R \) and under the graph of a nonnegative function \( f \) is called the double integral \( f \) over \( R \) and is denoted by \( \int \int_{R} f(x, y) \, dxdy \).

**Example 5.4.** \( f(x, y) = k, R = [0, 1] \times [0, 1], \int_{R} f(x, y) \, dxdy = k \).

**Theorem 5.5.**
(1) Any continuous function defined on a closed rectangle \( R \) is integrable.
(2) (Fubini Theorem) Let \( f \) be a continuous function defined on a \([a, b] \times [c, d]\). Then
\[\int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy = \int \int_{R} f(x, y) \, dA.\]

**Example 5.6.** \( f(x, y) = (x^2 + y), R = [0, 1] \times [0, 1]. \int_{R} f \, dA = \frac{5}{6} \). Use two different ways.
\[ f(x, y) = y(x^3 - 12x), R = [-2, 1] \times [0, 1]. \int_{R} f \, dA = \frac{57}{8}. \] (Fubini remains valid in the case \( f \) is negative or changes sign on \( R \))