1. (a) (5 points) (a) Find the equation of plane passing through the points \( P = (2, 0, 0) \), \( Q = (0, -1, 0) \), \( R = (0, 0, 3) \). (b) Find the distance from the origin to this plane.

First, find two vectors on plane:
\[ \vec{V}_1 = \overrightarrow{PQ} = (2, 0, 0) - (0, -1, 0) = (2, 1, 0) \]
\[ \vec{V}_2 = \overrightarrow{PQ} = (0, 1, 3) \]

Then, the normal \( \vec{N} = \vec{V}_1 \times \vec{V}_2 = (3, -6, 2) \)

The eqn of the plane is \( \vec{N} \cdot \vec{r} = 0 \), where \( \vec{r} \) is a vector on the plane.

So, \( \vec{N} = (x-x_0, y-y_0, z-z_0) \)

Take \((x_0, y_0, z_0) = P = (2, 0, 0) \) (can be \( Q \) or \( R \) as well)

And, dist = \[ \frac{|1A x_1 + by_1 + cz_1 + 0|}{||N||} \]

where \((x_1, y_1, z_1) = (0, 0, 0) \)

(b) (5 points) \[ \int_0^1 \int_0^y \int_0^{\sqrt{y^2+y^2}} 2x \, dz \, dx \, dy \]

\[ \frac{\pi}{12} \]

\[ = \int_0^y \arctan(\frac{z}{x}) \bigg|_0^y \, dx \, dy \]

\[ = \int_0^y \arctan(\frac{1}{y}) \, dx \, dy \]

\[ = \int_0^y \frac{1}{2} \, dx \, dy = \frac{1}{2} \int_0^y dy = \frac{y}{2} \]
(c) (5 points) Calculate \( \int_C (x+y+z) \, ds \), where \( C \) is the helix \( c(t) = (\cos t, \sin t, t) \) for \( 0 \leq t \leq 3\pi \).

\[ C'(t) = (-\sin t, \cos t, 1), \quad ||C'(t)|| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \]

\[ \int_C (x+y+z) \, ds = \int_0^{3\pi} (\cos t + \sin t + t) \cdot \sqrt{2} \, dt \]

\[ = \left[ 2\sqrt{2} + \frac{9\sqrt{2}}{2} \right] \pi^2 \]

(d) (5 points) Let \( C \) be a closed curve that is the boundary of a surface \( S \), calculate \( \int_C (1, 2, 3) \cdot ds \).

A: 0.

If \( f(x,y,z) = x+2y+3z \), then \( \nabla f = (1, 2, 3) \).

So, we are integrating the gradient field of \( f \).

Thus, \( \int_C (1, 2, 3) \cdot ds = f(C(3)) - f(C(0)) = f(3) - f(0) = 0 \).

Since \( C \) is a closed curve, \( f(3) = f(0) \).
2. (a) (10 points) Evaluate \( \iint_S F \cdot dS \), where \( F = (x, y, z) \), and \( S \) is the unit sphere \( x^2 + y^2 + z^2 = 1 \).

Two options

(i) Parametrize the sphere

\[
F(x,y,z) = (\sin \cos \psi, \sin \sin \psi, \cos \psi)
\]

\( \theta, \psi \in [0, \pi], \phi \in [0, 2\pi] \)

Then,

\[
T_\theta = (\cos \cos \psi, \sin \sin \psi, -\sin \psi), \quad T_\psi = (-\sin \sin \psi, \sin \cos \psi, 0)
\]

And,

\[
T_\theta \times T_\psi = (\sin \cos \cos \psi, \sin \sin \sin \psi, \sin \cos \sin \psi)
\]

\*Note \( T_\theta \times T_\psi \) is oriented outward.

(in particular, \( T_\theta \times T_\psi \) points in the direction of radial vector.)

So,

\[
\iint_S F \cdot dS = \iiint (\sin \cos \psi, \sin \sin \psi, \cos \psi) \cdot (\sin \cos \cos \psi, \sin \sin \sin \psi, \sin \cos \sin \psi) \sin \psi d\theta d\psi
\]

= \boxed{4\pi}

(b) (10 points) Let \( D \) be the unit disc, compute \( \int_{\partial D} P \, dx \), where \( P = \frac{y}{x^2 + y^2} \), and \( \partial D \) is the oriented boundary of \( D \).

\( D \) is unit disc, \( \partial D \) is the boundary (circle)

It is oriented counterclockwise since \( D \) should be to the left as you walk around.

\[
\int_{\partial D} P \, dx = \int_0^{2\pi} P(t) \times x'(t) \, dt = \int_0^{2\pi} \frac{\sin t}{\cos^2 t + \sin^2 t} \cdot -\sin t \, dt
\]

\[
= \int_0^{2\pi} -\sin^2 t \, dt = \frac{1}{2} \int_0^{2\pi} (\cos 2t - 1) \, dt = \frac{1}{2} (\frac{1}{2} \sin 2t - 2t)
\]

= \boxed{-\pi}
3. (20 points) State Stoke's theorem (5 pts). Use Stoke's theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{s} \), where \( C \) is the triangle with vertices \((1, 0, 0), (0, 1, 0), (0, 0, 1)\) oriented counterclockwise when viewed from above and \( \mathbf{F} = (x + y^2, y + z^2, z + x^2) \). (15 pts)

A: -1.

Note, \( \text{curl} \mathbf{F} = \mathbf{F} \times \mathbf{F} = (-2z, -2x, -2y) \)

The curve \( C \) is \( \gamma \)

So \( S \) (where \( \partial S = C \))

\( S \) lies on the plane \( x + y + z = 1 \), so we can parameterize as a graph

\[ z = 1 - x - y \Rightarrow \mathbf{r}(u, v) = (u, v, 1 - u - v) \]

\[ 0 \leq u \leq 1, \quad 0 \leq v \leq 1 - u \]

Then, \( T_u = (1, 0, -1), \quad T_v = (0, 1, -1), \quad T_u \times T_v = (1, 1, 1) \) and this is in the direction of induced normal normal

\[ \int_C \mathbf{F} \cdot d\mathbf{s} = \int_S \text{curl} \mathbf{F} \cdot d\mathbf{s} \]

\[ = \int_0^1 \int_0^{1-u} \left( -2(1 - u - v) - 2u - 2v \right) \cdot (1, 1, 1) \, dv \, du \]

\[ = \int_0^1 \int_0^{1-u} -2 \, dv \, du = \left[ -1 \right] \]

Alternatively, on \( x + y + z = 1 \), \((1, 1, 1)\) is normal to the plane, so

\[ \mathbf{n} = \frac{(1, 1, 1)}{\sqrt{3}} \]

is the unit normal (oriented)

\[ \frac{11 \sqrt{3}}{4} x + y + z = 1 \]

So,

\[ \int_C \mathbf{F} \cdot d\mathbf{s} = \int_S \text{curl} \mathbf{F} \cdot \mathbf{n} \, dS = \int_S \frac{-2}{\sqrt{3}} \left( x + y + z \right) 

\[ = \frac{-3}{\sqrt{3}} \int_S 1 \, dS = \frac{3}{\sqrt{3}} \text{Area}(\text{triangle}) = -1 \]
4. (20 points) Let $C^+$ be the perimeter of the square $[0, 1] \times [0, 1]$ in the counterclockwise direction. Evaluate the line integral $\int_{C^+} x^2 \, dx + xy \, dy$.

We may apply Green's Theorem to this line integral:

\[ \int_{C^+} x^2 \, dx + xy \, dy = \iint_D \left( \frac{\partial}{\partial y} (xy) - \frac{\partial}{\partial x} (x^2) \right) \, dA \]

\[ = \iint_D y - 0 \, dA = \int_0^1 \int_0^1 y \, dy = \frac{1}{2} \]
5. (20 points) The intersection of the plane \( x + \frac{1}{2} y + \frac{1}{3} z = 0 \) with the unit sphere \( x^2 + y^2 + z^2 = 1 \) is a great circle. Use Lagrange Multipliers to find the point on this great circle with the greatest \( x \)-coordinate.

We have two constraints:

\[
\begin{align*}
G_1 &= x + \frac{1}{2}y + \frac{1}{3}z = 0 \\
G_2 &= x^2 + y^2 + z^2 = 1
\end{align*}
\]

We maximize: \( f(x,y,z) = y \)

Then there is \( \lambda_1, \lambda_2 \) so:

\[
\nabla f = \lambda_1 \nabla G_1 + \lambda_2 \nabla G_2
\]

\[
\begin{align*}
\left( \begin{array}{c}
1 \\
0 \\
0
\end{array} \right) &= \lambda_1 \left( \begin{array}{c}
\frac{1}{2} \\
\frac{1}{3}
\end{array} \right) + \lambda_2 \left( \begin{array}{c}
x \\
y \\
z
\end{array} \right)
\end{align*}
\]

A system of

3 eqns:

\[
\begin{align*}
(1) \quad 1 &= \lambda_1 + 2\lambda_2 x \\
(2) \quad 0 &= \frac{1}{2} \lambda_1 + 2\lambda_2 y \\
(3) \quad 0 &= \frac{1}{3} \lambda_2 + 2\lambda_2 z
\end{align*}
\]

Adding \((1) + \frac{1}{2}(2) + \frac{1}{3}(3)\):

\[
1 = \lambda_1 \left(1 + \frac{1}{4} + \frac{1}{9} \right) + 2\lambda_2 \left(x + \frac{1}{3}y + \frac{1}{3}z \right)
\]

\[
\left[ \begin{array}{c}
\lambda_1 = \frac{36}{26} \\
\lambda_2 = \frac{26}{26}
\end{array} \right]
\]

\[
\left[ \begin{array}{c}
\frac{36}{26} x^2 + (-2\sqrt{2}) x (-2\sqrt{2}) y + (-2\sqrt{2})^2 z = 1 \\
\frac{26}{26} (x^2 + y^2 + z^2) = 1
\end{array} \right]
\]

\[
9 = \frac{26}{26} x^2 + \frac{26}{26} y^2 + \frac{26}{26} z^2
\]

\[
(1) \quad x = 13 \\
(2) \quad y = -18 \\
(3) \quad z = -16
\]

So, \( x = \pm \frac{13}{\sqrt{13^2 + 18^2 + 16^2}} \)

So \( x = 13 \)

So \( y = -18 \)

So \( z = -16 \)

The max must be \( x = \frac{13}{\sqrt{13^2 + 18^2 + 16^2}} \)

So \( y = -\frac{18}{\sqrt{13^2 + 18^2 + 16^2}} \)

So \( z = -\frac{16}{\sqrt{13^2 + 18^2 + 16^2}} \)