2) Sometimes it is possible to do these problems by just fiddling around with inequalities instead of worrying about geometry. This is one of those cases. Notice that the bounds of our integrals are telling us that we have $0 \leq y \leq 1$ and $y \leq x \leq 1$. These inequalities combine nicely to yield the equivalent inequality $0 \leq y \leq x \leq 1$. This compound inequality can now be broken back up into $0 \leq x \leq 1$ and $0 \leq y \leq x$, which will be our new bounds of integration. This works because the pairs $(x, y)$ that obey the starting pair of inequalities are the same as the ones that obey the final pair of inequalities.

**Careful:** This technique will not always work. For example, if we had $0 \leq y \leq 2$ and $y \leq x \leq 1$, we couldn’t combine them to form a compound inequality $0 \leq y \leq x \leq 1$, because in the compound inequality $y$ runs from 0 to 1, whereas originally it was from 0 to 2. Anyway, finishing our original computation,

$$
\int_0^1 \int_0^x \sin(x^2)dydx = \int_0^1 x \sin(x^2)dx = -\frac{1}{2} \cos(x^2) \bigg|_0^1 = \frac{1}{2} \left(1 - \cos(1)\right)
$$

4a) Manipulating inequalities won’t work as well here, so let’s look at the region in question. We have a triangle that is being cut into horizontal strips.\(^1\) We will change the order of integration to do this problem.\(^2\)

If we chop this region into vertical strips, the first strip will be along $x = 0$ (though this ”strip” consists of only one point, but whatever), and the last strip is along $x = 1$. For a strip with some particular $x$ coordinate, the strip goes from $y = -x$ to $y = x$. Thus,

$$
\int_0^1 \int_{-x}^x x^2 + 2xy + y^2 dydx = \frac{2}{3}
$$

\(^1\)Which we know because the outer integral says we are going between the lines $y = -1$ and $y = 1$.

\(^2\)It is arguably easier to evaluate the integral without changing the order of integration, but let’s do it anyway for the practice.
4b) In this case the region is a circle of radius 3 with the top missing. It is being cut into horizontal strips:

![Diagram of a circle with a top missing and horizontal strips](image)

The important moral of this question is that you don’t always want to change the order of integration. In this case, horizontal strips are already the way to go, and trying to use vertical strips instead is fairly blegh. If you cut the region into vertical strips and look at said strips, their upper bound changes whenever you pass over one of the dashed lines. This leads to casework, which we’d rather avoid. To top it all off, the integral we get is still annoying to compute! So, we bash the starting integral.

\[
\int_{-3}^{1} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 \, dx \, dy = \int_{-3}^{1} \frac{2}{3} (9 - y^2)^{3/2} \, dy
\]

This looks like a job for trig substitution! Let \( y = 3 \sin(\theta), \, dy = 3 \cos(\theta) \, d\theta \):

\[
2 \int_{-3}^{1} (9 \cos^2(\theta))^{3/2} \cos(\theta) \, d\theta = 54 \int_{-3}^{1} \cos^3(\theta) \, d\theta = \frac{3}{2} + \frac{1}{4} \sin(2) + \frac{1}{32} \sin(4) + \frac{1}{4} \sin(6) + \frac{1}{32} \sin(12) \approx 1.6171
\]

Note that to compute the integral, we make heavy use of the trig double angle formulas, such as \( \cos^2(\theta) = (1/2)(1 + \cos(2\theta)) \).

4c) This is another example of where we can just mess with inequalities! Our integral bounds tell us \( 0 \leq y \leq 4 \) and \( y/2 \leq x \leq 2 \). Doubling the second inequality yields \( y \leq 2x \leq 4 \), so our inequalities combine to give \( 0 \leq y \leq 2x \leq 4 \). This breaks apart as \( 0 \leq 2x \leq 4 \), and \( 0 \leq y \leq 2x \). So,

\[
\int_{0}^{2} \int_{0}^{2x} e^{x^2} \, dy \, dx = \int_{0}^{1} 2xe^{x^2} \, dx = e - 1
\]

4d) We have the inequalities \( 0 \leq y \leq 1 \) and \( \arctan(y) \leq x \leq \pi/4 \). Since \( \tan \) is a monotonically increasing function, we can take \( \tan \) of our second inequality to get \( y \leq \tan(x) \leq 1 \). So, once again the problem has been rigged so we can combine the inequalities and get \( 0 \leq y \leq \tan(x) \leq 1 \). We can break this back apart as \( 0 \leq \tan(x) \leq 1 \) (aka \( 0 \leq x \leq \pi/4 \)) and \( 0 \leq y \leq \tan(x) \).

\[
\int_{0}^{\pi/4} \int_{0}^{\tan(x)} \sec^5(x) \, dy \, dx = \int_{0}^{\pi/4} \tan(x) \sec^5(x) \, dx
\]

Substitute \( u = \sec(x), \, du = \sec(x) \tan(x) \, dx \):

\[
\int_{1}^{\sqrt{2}} x^4 \, dx = \frac{4\sqrt{2} - 1}{5}
\]
9) Rewriting our integrand as \( f(x, y) = \frac{1}{x^2 + 1} \), it is obvious that \( f \) takes on its max at \( f(0, 0) = 1 \), and its min at \( f(1, 2) = \frac{1}{6} \). The area of our region of integration is 6, so the mean value inequality gives us exactly the inequality we wanted.

**Section 5.5**

2) By writing the integral as \( \int \int \int \sin(x) dx dy dz \), they are trying to trick you into writing down

\[
\int_0^x \int_0^1 \int_0^\pi \sin(x) dx dy dz
\]

However, this doesn’t mean anything! You are letting \( x \) go between 1 and \( \pi \) in the innermost integral, so it doesn’t make sense to to then have an \( x \) outside it. There are a few orderings we can use here. One of them is

\[
\int_0^\pi \int_0^1 \int_0^x \sin(x) dz dy dx = \pi
\]

3)

\[
\int_0^1 \int_0^1 \int_0^1 x^2 dx dy dz = \frac{1}{3}
\]

7) This is a problem on 3D elementary regions, as described on p.297 of the book. This region can be visualized by first drawing the graphs of \( |x| \) and \( x^2 \) and then rotating everything around the vertical axis:

![Visual Diagram](image)

The result is a cone inside of a paraboloid. The elementary description is as follows. The region is all points above the unit disk \( x^2 + y^2 \leq 1 \) such that their \( z \) coordinates obey \( x^2 + y^2 \leq z \leq \sqrt{x^2 + y^2} \).

There are a few ways to know we are above the unit disk. One is to note that there are only values of \( z \) that obey the inequality \( x^2 + y^2 \leq z \leq \sqrt{x^2 + y^2} \).
The region in consideration is a tetrahedron with one vertex at the origin, and the others a distance $a$ away along the axes:

Note that this region is elementary. It is all points above the triangle $T$ in the $xy$ plane bounded by $x = 0, y = 0, x + y = a$, and such that the $z$ coordinates obey $0 \leq z \leq a - x - y$. We chop up the 2D region $T$ in the obvious way, yielding

$$\int_0^a \int_0^{a-x} \int_0^{a-x-y} (x^2 + y^2 + z^2)dzdydx = \frac{a^5}{20}$$

We draw the region in two steps. The first three constraints give us a triangular prism that extends upwards infinitely, but is stopped below by the plane $z = 0$. This is the first picture. The plane $z = x + y$ cuts through the three points $(0,0,0)$ $(1,0,1)$ $(0,1,1)$, which are labeled in the first picture. Adding this plane to the second picture, we can see that the desired region is a tetrahedron.

This tetrahedron is an elementary region. It lives above the triangle $T$ with sides $x = 0, y = 0, x + y = 1$, and has $z$ coordinates constrained by $0 \leq z \leq x + y$. To find the volume of a region, you integrate the constant function $1$ over that region:

$$\text{volume} = \int_0^1 \int_0^{1-x} \int_0^{x+y} 1 \, dzdydx = \frac{1}{3}$$

$$\int_0^1 \int_0^{1-x} \int_0^{x+y} x \, dzdydx = \frac{1}{8}$$

$$\int_0^1 \int_0^{1-x} \int_0^{x+y} y \, dzdydx = \frac{1}{8}$$
1a) \( T \) is 1 to 1: If \( T(x_1, y_1) = T(x_2, y_2) \), then \( (2x_1, y_1) = (2x_2, y_2) \), and so we conclude \( x_1 = x_2, y_1 = y_2 \). 

\( T \) is onto: Given a point \((u, v)\) in the range, obviously \( T(u/2, v) = (u, v) \).

1b) \( T \) is not 1 to 1: \( T(1, 0) = T(-1, 0) \).

1c) \( T \) is 1 to 1: If \( T(x_1, y_1) = T(x_2, y_2) \), then \( (\sqrt[3]{x_1}, \sqrt[3]{y_1}) = (\sqrt[3]{x_2}, \sqrt[3]{y_2}) \), and so \( x_1 = x_2, y_1 = y_2 \).

\( T \) is onto: Given a point \((u, v)\) in the range, \( T(u^3, v^3) = (u, v) \).

1d) \( T \) is not 1 to 1: \( T(0, 0) = T(\pi, 0) \).

4) A linear map is just a matrix, so say our map is given by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). One thing you need to know about linear maps is that they send parallelograms to parallelograms (note that a rectangle is a parallelogram), and in particular the vertices of the starting parallelogram will become the vertices of the ending parallelogram. Further, for any linear transformation, \( T(0, 0) = (0, 0) \), so our transformation will definitely send one vertex of \( D^* \) to one vertex of \( D \).

We now pick the variables \((a, b, c, d)\) so that \( T(1, 0) = (-1, 3) \), and \( T(0, 1) = (-1, -3) \). Then, \( T \) will send three of the vertices of \( D^* \) to three of the vertices of \( D \). Since \( T(D^*) \) will be a parallelogram, three of whose vertices match those of \( D \), it follows that \( T(D^*) = D \). Finally, we compute:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}
\]

Solving the resulting equations yields \( a = -1, c = 3, b = -1, d = -3 \).

9) We start by looking at what happens to the edges of the square. The bottom edge (call it a) can be parametrized as the points \((t, 0), 0 \leq t \leq 1\). Plugging in, \( T(a) \) is \((0, t), 0 \leq t \leq 1\), which is a vertical segment. We repeat for the other sides of the square and get:

Note that the side d gets collapsed to a point. It should now be clear that \( T(D^*) \) is the enclosed triangular region.

\( T \) is not 1 to 1: \( T(0, 0) = T(0, 1) \). However, let’s look at how much 1 to 1ness fails. Say \( T(x_1, y_1) = T(x_2, y_2) \). Then, \( (x_1y_1, x_1) = (x_2y_2, x_2) \). Obviously, \( x_1 = x_2 \). If \( x_1 = x_2 \neq 0 \), then we can also conclude \( y_1 = y_2 \). Thus, \( T(x, y) \) is 1 to 1, as long as \( x \neq 0 \). We can choose to eliminate all points of the edge d, except for \((0,0)\).
12) These last two problems are quite tricky if you don’t know linear algebra, but if you do they are fairly trivial results. The real takeaway should be the relationship between the determinant and 1 to 1/onto-ness for linear transformations. The proofs themselves (assuming you don’t know lin alg) are kinda icky and not worth stressing over.

Let $T$ be given by \[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
\] Assume $T$ is one to one. Then, $T(0,0) \neq T(b,-a)$. Expanding, we get $(0,0) \neq (0, cb - ad)$. Thus, $ad - bc \neq 0$.

For the other direction, assume $ad - bc \neq 0$. To prove $T$ is 1 to 1, say that $T(x_1, y_1) = T(x_2, y_2)$. Then, \[
(ax_1 + by_1, cx_1 + dy_1) = (ax_2 + by_2, cx_2 + dy_2).
\] This gives us two equations, which rearrange to
\[a(x_1 - x_2) + b(y_1 - y_2) = 0, 
\] and
\[c(x_1 - x_2) + d(y_1 - y_2) = 0.\]

If we take $d$ times the first equation, and subtract $b$ times the second equation, the $y$ terms cancel and we get $(ad - bc)(x_1 - x_2) = 0$. Since $ad - bc \neq 0$, we conclude $x_1 = x_2$. In a similar way, we may cancel the $x$ terms and get $y_1 = y_2$.

13) Assume $ad - bc = 0$. Then, $T(x, y) = (ax + by, cx + dy)$. Notice that $c(ax + by) - a(cx + dy) = (ca - ac)x + (cb - ad)y = 0$. Thus, we can only hit points $(u, v)$ in the range such that $cu - dv = 0$. So, in this case $T$ is not onto.\(^3\)

For the other direction, assume $ad - bc \neq 0$. Let $(u, v)$ be some point in the range. We want to find $(x, y)$ so $T(x, y) = (u, v)$. This gives us two equations to solve:
\[ax + by = u, \ cx + dy = v\]

Taking $d$ times the first equation and subtracting $b$ times the second gives $(ad - bc)x = du - bv$, and so $x = \frac{du - bv}{ad - bc}$. Using a similar approach, we can solve for $y$. This proves that $(u, v)$ is struck, since we found a pair $(x, y)$ that hits it.

---

\(^3\)To conclude the proof, notice that $(u, v) = (c, -d)$ cannot be hit, unless $c = d = 0$, but in that case it is easy to prove directly $T$ is not onto.