1. Points would only be awarded for correct answers. There is no partial credit for any incorrect one.

(a) Compute the arc length of \( c(t) = (x(t), y(t)) = (|t|, |t - \frac{1}{2}|), -1 \leq t \leq 1. \)

When \( t \leq 0 \), \( c(t) = (-t, \frac{1}{2} - t) \), when \( 0 \leq t \leq \frac{1}{2} \), \( c(t) = (t, \frac{1}{2} - t) \), when \( 1 \geq t \geq \frac{1}{2} \), \( c(t) = (t, t - \frac{1}{2}) \). The arc length is

\[
\int_{-1}^{0} ||c'(t)||dt + \int_{\frac{1}{2}}^{1} ||c'(t)||dt + \int_{\frac{1}{2}}^{1} ||c'(t)||dt \\
= \int_{-1}^{0} \sqrt{1 + 1} dt + \int_{0}^{\frac{1}{2}} \sqrt{1 + 1} dt + \int_{\frac{1}{2}}^{1} \sqrt{1 + 1} dt \\
= 2\sqrt{2}.
\]

(b) Let \( F(x, y, z) = (x, y, z) \). Find a function \( f : 3 \rightarrow \) such that \( F = \nabla f \).

Let \( f = \frac{1}{2}(x^2 + y^2 + z^2) \), \( \nabla f = (x, y, z) \).

(c) Compute the curl of \( V(x, y) = yi - xj \).

\[
\left| \begin{array}{ccc}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & 1 & 0 \\
-x & 0 & 1
\end{array} \right| = \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right)i + \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial x} \right)j + \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)k = -2k.
\]

(d) Let \( F = (xy, yz, zx) \), compute \( \nabla \cdot (\nabla \times F) \).

\[
\nabla \times F = \left| \begin{array}{ccc}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x & y & z \\
y & z & x
\end{array} \right| = -yi - zj - xk.
\]

\[
\nabla \cdot (\nabla \times F) = \nabla \cdot (-yi - zj - xk) = 0.
\]

Remark: the divergence of any curl of \( F \) is 0.
2. Find the volume of the region which is defined by
\[ \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1, x \leq y + z \leq 1\} \]

Correct bounds for the region

\[ W : 0 \leq x \leq 1 \]
\[ 0 \leq y \leq 1 - x \]
\[ 0 \leq z \leq 1 - x - y \]

\[ \text{Vol}(W) = \iiint_W 1 \, dz \, dy \, dx = \int_0^1 \int_{0}^{1-x} \int_{0}^{1-x-y} 1 \, dz \, dy \, dx \]
\[ = \int_0^1 \int_{0}^{1-x} (1-x-y) \, dy \, dx \]
\[ = \int_0^1 \left[ (1-x)y - \frac{1}{2}y^2 \right]_0^{1-x} \, dx \]
\[ = \int_0^1 \frac{1}{2} (1-x)^2 \, dx \]
\[ = \left[ -\frac{1}{6} (1-x)^3 \right]_0^1 \]
\[ = \frac{1}{6} \]

* 15 points were given for the right bounds and 5 points for the computation. There were partial points for incorrect bounds.

Examples for 10 points:
\[ \int_0^1 \int_0^{1-x} \int_{0}^{1-x-y} dz \, dy \, dx, \int_0^1 \int_{0}^{1-x} \int_0^{1-x-y} dz \, dy \, dx, \int_0^1 \int_{0}^{1-x} \int_0^{1-x-y} dz \, dy \, dx \]

Examples for 5 points:
\[ \int_0^1 \int_0^{1-x} \int_{0}^{1-x-y} (x+y+z) \, dz \, dy \, dx, \int_0^1 \int_{0}^{1-x} \int_0^{1-x-y} dz \, dy \, dx, \int_0^1 \int_{0}^{1-x} \int_0^{1-x-y} dz \, dy \, dx, \int_0^1 \int_0^{1-x} \int_{0}^{1-x-y} dz \, dy \, dx, \ldots \] and so on.
We want to compute $\int\int_{D_\alpha} e^{-(x^2+y^2)} \, dx \, dy$ where $D_\alpha$ is the disc of radius $a$. To compute this we will change to polar coordinates.

We substitute $r^2 = x^2 + y^2$ and multiply the integrand by $\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = r$ to get

$$\int\int e^{-r^2} r \, dr \, d\theta$$

Now, $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq a$, so we can compute with the bounds:

$$\int\int_{D_\alpha} e^{-(x^2+y^2)} \, dx \, dy = \int_0^{2\pi} \int_0^a e^{-r^2} r \, dr \, d\theta$$

$$= \frac{-1}{2} \int_0^{2\pi} e^{-a^2} - \int_0^a 0 \, d\theta$$

$$= \frac{-1}{2} \int_0^{2\pi} e^{-a^2} - 1 \, d\theta$$

$$= \pi (1 - e^{-a^2})$$

Rubric: 5 points were given for putting the polar coordinate change of variable factor $r$, and 5 points were given for converting $e^{-(x^2+y^2)}$ as $e^{-r^2}$. The rest of the points were given for getting the correct bounds on the polar coordinate region and correctly evaluating the integral.
The region is bound by the graph of \( y = \sqrt{x} \), the y-axis, and the line \( y = 1 \). (5 pts)

We have

\[
\int_0^1 \int_{\sqrt{x}}^1 \frac{1}{\sqrt{1 + y^3}} \, dy \, dx = \int_0^1 \int_0^y \frac{1}{\sqrt{1 + y^3}} \, dx \, dy
\]

(7 pts)

\[
= \int_0^1 \frac{y^2}{\sqrt{1 + y^3}} \, dy
\]

(2 pts)

\[
= \frac{1}{3} \int_0^1 \frac{u}{\sqrt{1 + u}} \, du
\]

(2 pts)

\[
= \frac{2}{3} \sqrt{1 + u} \bigg|_0^1
\]

(2 pts)

\[
= \frac{2}{3} (\sqrt{2} - 1)
\]

(2 pts)
There are three steps to changing coordinates. You need to convert the region into $uv$ coordinates (6 points), you need to convert the function into $uv$ coordinates (4 points), and then you need a Jacobian (8 points). Finally, you need to evaluate the $uv$ integral (2 points).

We begin with the region. Some quick algebra/geometry shows that our region is $0 \leq x - 2y \leq 3$, $-3 \leq y - 2x \leq 0$. Since $u = x - 2y, v = y - 2x$, in $uv$ coordinates our region is simply the rectangle $0 \leq u \leq 3, -3 \leq v \leq 0$.

Next we rewrite the function $f(x, y) = xy$ in $uv$ coordinates. We have to solve for $x$ and $y$ in terms of $u$ and $v$. This is just some linear algebra, and the right answer is $x = -(1/3)(u + 2v), y = -(1/3)(2u + v)$. So, plugging in, we get that $xy = (1/9)(u + 2v)(2u + v)$.

For the Jacobian, we need to compute $\frac{\partial x,y}{\partial u,v}$. Since we just computed $x$ and $y$ in terms of $u$ and $v$, we can use these expressions to get the partial derivatives we need:

\[
\frac{\partial x,y}{\partial u,v} = \begin{vmatrix} -1/3 & -2/3 \\ -2/3 & -1/3 \end{vmatrix} = 1/9 - 4/9 = -1/3
\]

However, do not forget that the Jacobian is the absolute value of this determinant. So, the Jacobian is $1/3$.

Our integral is $uv$ coordinates is then

\[
\frac{1}{27} \int_{0}^{3} \int_{-3}^{0} (u + 2v)(2u + v) \, dv \, du = \frac{1}{4}
\]