The Alexander–Whitney chain map

The formula There is a well-known natural transformation α, which consists of homomorphisms

$$\alpha(X,Y)\colon C(X\times Y)\longrightarrow C(X)\otimes C(Y)$$

for all spaces X and Y, where $C(X)$ denotes the singular chain complex of X. In degree n, it is defined on any singular simplex $\sigma\colon\Delta^n\to X\times Y$, with coordinates $\sigma_1\colon\Delta^n\to X$ and $\sigma_2\colon\Delta^n\to Y$, by the *Alexander–Whitney* formula:

$$\alpha_n\sigma=\sum_{k+l=n}\sigma_1\circ\lambda^n_k\otimes\sigma_2\circ\rho^n_l\qquad\text{in } C(X)\otimes C(Y),\tag{1}$$

where $\lambda^n_k\colon\Delta^k\subset\Delta^n$ and $\rho^n_l\colon\Delta^l\subset\Delta^n$ are the linear maps defined on the vertices by $\lambda^n_k(e_i)=e_i$ (for $0\le i\le k$) and $\rho^n_l(e_i)=e_{n-l+i}$ (for $0\le i\le l$). The map λ^n_k embeds Δ^k as the "lowest" k-dimensional face of Δ^n and ρ^n_l embeds Δ^l as the "highest" l-dimensional face of Δ^n.

PROPOSITION 2 *The homomorphisms α_n do in fact form an augmented chain map.*

Iterated face maps We can easily express the maps λ^n_k and ρ^n_l in terms of the standard face inclusions $\eta_i\colon\Delta^n\subset\Delta^{n+1}$ (for $0\le i\le n+1$), where η_i omits the vertex e_i from its image, as follows:

$$\lambda^n_k=\eta_n\circ\eta_{n-1}\circ\ldots\circ\eta_{k+2}\circ\eta_{k+1}\colon\Delta^k\subset\Delta^n$$

and

$$\rho^n_l=\eta_0\circ\eta_0\circ\ldots\circ\eta_0\colon\Delta^l\subset\Delta^n\qquad\text{(with } n-l \text{ factors)}.$$

We need to know how they interact with the maps η_i. [The following results may also be proved by induction, using the standard formula

$$\eta_i\circ\eta_j=\eta_j\circ\eta_{i-1}\colon\Delta^n\subset\Delta^{n+2}\quad\text{for } 0\le j<i\le n+2.\tag{3}$$

Both these composites omit e_j and e_i from their images.]

LEMMA 4 *We have the following relations:*

(a) *For maps $\Delta^k\to\Delta^{n+1}$ involving λ:*

$$\eta_i\circ\lambda^n_k=\begin{cases}\lambda^{n+1}_{k+1}\circ\eta_i, & \text{if } i\le k;\\ \lambda^{n+1}_k, & \text{if } i>k.\end{cases}\tag{5}$$

and

$$\lambda^{n+1}_{k+1}\circ\eta_{k+1}=\lambda^{n+1}_k.\tag{6}$$

(b) *For maps $\Delta^l\to\Delta^{n+1}$ involving ρ:*

$$\eta_i\circ\rho^n_l=\begin{cases}\rho^{n+1}_l, & \text{if } i\le n-l;\\ \rho^{n+1}_{l+1}\circ\eta_{i-n+l}, & \text{if } i>n-l.\end{cases}\tag{7}$$

and

$$\rho^{n+1}_{l+1}\circ\eta_0=\rho^{n+1}_l.\tag{8}$$

Proof In (a), (5) is clear for $i > k$, as the image of λ_k^n is spanned by $e_0, e_1, \dots e_k$, which does not include e_i. If $i \le k$, the image of $\eta_i \circ \lambda_k^n$ omits e_i and is spanned by $e_0, e_1, \dots, e_{i-1}, e_{i+1}, \dots e_{k+1}$; we get the same effect by first applying $\eta_i \colon \Delta^k \to \Delta^{k+1}$ and then λ_{k+1}^{n+1}. In (6), η_{k+1} leaves $e_0, e_1, \dots, e_k$ fixed, and so do λ_{k+1}^{n+1} and λ_k^{n+1}.

Part (b) can be proved analogously. Alternatively, it is useful to introduce the *inversion* linear map $u_n \colon \Delta^n \to \Delta^n$ (not part of the standard simplicial structure) given by $u_n(e_i) = e_{n-i}$ for all i. We observe that $\rho_l^n = u_n \circ \lambda_l^n \circ u_l$, $u_{n+1} \circ \eta_i \circ u_n = \eta_{n+1-i}$, and $u_n \circ u_n = \mathrm{id}$; then (b) follows immediately from (a). □

Proof of the chain map

Proof of Proposition 2 Given a singular simplex $\tau \colon \Delta^{n+1} \to X \times Y$, with coordinates $\tau_1 \colon \Delta^{n+1} \to X$ and $\tau_2 \colon \Delta^{n+1} \to Y$, we have to show that $\partial \alpha_{n+1}(X,Y)\,\tau = \alpha_n(X,Y)\,\partial\tau$.

From (1),

$$\begin{aligned}
\partial\alpha_{n+1}\tau &= \sum_{r+s=n+1} \partial(\tau_1 \circ \lambda_r^{n+1} \otimes \tau_2 \circ \rho_s^{n+1}) \\
&= \sum_{r+s=n+1} \partial(\tau_1 \circ \lambda_r^{n+1}) \otimes \tau_2 \circ \rho_s^{n+1} + \sum_{r+s=n+1} (-1)^r\, \tau_1 \circ \lambda_r^{n+1} \otimes \partial(\tau_2 \circ \rho_s^{n+1}) \\
&= \sum_{r+s=n+1} \sum_{i=0}^{r} (-1)^i\, \tau_1 \circ \lambda_r^{n+1} \circ \eta_i \otimes \tau_2 \circ \rho_s^{n+1} \\
&\qquad + \sum_{r+s=n+1} \sum_{j=0}^{s} (-1)^{r+j}\, \tau_1 \circ \lambda_r^{n+1} \otimes \tau_2 \circ \rho_s^{n+1} \circ \eta_j
\end{aligned}$$

and

$$\alpha_n \partial\tau = \sum_{m=0}^{n+1} (-1)^m \alpha_n(\tau \circ \eta_m) = \sum_{m=0}^{n+1} \sum_{r+s=n} (-1)^m\, \tau_1 \circ \eta_m \circ \lambda_r^n \otimes \tau_2 \circ \eta_m \circ \rho_s^n.$$

We choose k and l such that $k + l = n$ and pick out the terms that lie in $C_k(X) \otimes C_l(Y)$. In $\partial\alpha_{n+1}\tau$ we find

$$\sum_{i=0}^{k+1} (-1)^i\, U_i + \sum_{j=0}^{l+1} (-1)^{k+j}\, V_j,$$

where $U_i = \tau_1 \circ \lambda_{k+1}^{n+1} \circ \eta_i \otimes \tau_2 \circ \rho_l^{n+1}$ and $V_j = \tau_1 \circ \lambda_k^{n+1} \otimes \tau_2 \circ \rho_{l+1}^{n+1} \circ \eta_j$. In $\alpha_n\partial\tau$ we find $\sum_{m=0}^{n+1} (-1)^m\, W_m$, where $W_m = \tau_1 \circ \eta_m \circ \lambda_k^n \otimes \tau_2 \circ \eta_m \circ \rho_l^n$.

We note that $\partial\alpha_{n+1}\tau$ provides $k + l + 4 = n + 4$ terms, while $\alpha_n\partial\tau$ provides only $n + 2$ terms. Equations (5) and (7) show that $U_i = W_i$ for $0 \le i \le k$, where U_i and W_i have the same sign. These equations also show that $V_j = W_{k+j}$ for $1 \le j \le l + 1$, again with the same sign. The two remaining terms are U_{k+1} and V_0; (6) and (8) show that $U_{k+1} = V_0$, and these terms appear with opposite signs and so cancel.

For $n = 0$, (1) reduces to $\alpha_0\tau = \tau_1 \otimes \tau_2$, which obviously preserves the augmentation. □

Applications We deduce the derivation formula for the cup product,

$$\delta(\phi \cup \psi) = (\delta\phi) \cup \psi + (-1)^k \phi \cup (\delta\psi) \qquad \text{in } C^{k+l+1}(X;R), \tag{9}$$

where $\phi \in C^k(X;R)$, $\psi \in C^l(X;R)$ and R is any coefficient ring. Equivalently, we have the derivation formula for the cross product,

$$\delta(\phi \times \psi) = (\delta\phi) \times \psi + (-1)^k \phi \times \delta\psi \qquad \text{in } C^{k+l+1}(X \times Y;R), \tag{10}$$

where $\phi \in C^k(X;R)$ and $\psi \in C^l(Y;R)$. (In fact, either formula is essentially equivalent to Proposition 2.)

Let us abbreviate (1) to $\alpha_n\sigma = \sum_k \sigma'_k \otimes \sigma''_{n-k}$, where $\sigma'_k \in C_k(X)$ and $\sigma''_{n-k} \in C_{n-k}(Y)$. Then given $\phi \in C^k(X;R)$ and $\psi \in C^l(Y;R)$, their cross product $\phi \times \psi$ is defined on a singular $(k+l)$-simplex $\sigma\colon \Delta^{k+l} \to X \times Y$ as

$$\langle \phi \times \psi, \sigma \rangle = \langle \phi, \sigma'_k \rangle \langle \psi, \sigma''_l \rangle. \tag{11}$$

Since α is a chain map, we have the commutative diagram

$$\begin{array}{ccc} C_{k+l+1}(X \times Y) & \xrightarrow{\alpha_{k+l+1}} & C(X) \otimes C(Y) \\ \downarrow \partial & & \downarrow \partial \\ C_{k+l}(X \times Y) & \xrightarrow{\alpha_{k+l}} & C(X) \otimes C(Y) \end{array}$$

which we evaluate on a singular $(k+l+1)$-simplex $\tau\colon \Delta^{k+l+1} \to X \times Y$, and then apply $\langle \phi, - \rangle\colon C_k(X) \to R$ and $\langle \psi, - \rangle\colon C_l(Y) \to R$ and multiply. The lower route yields $\langle \phi \times \psi, \partial\tau \rangle = \langle \delta(\phi \times \psi), \tau \rangle$. If we take the upper route and recall the boundary ∂ in $C(X) \otimes C(Y)$, the resulting terms that lie in $C_k(X) \otimes C_l(Y)$ are

$$\partial\tau'_{k+1} \otimes \tau''_l + (-1)^k \tau'_k \otimes \partial\tau''_{l+1}.$$

We apply ϕ and ψ and multiply to obtain

$$\begin{aligned} \langle \phi, \partial\tau'_{k+1} \rangle \langle \psi, \tau''_l \rangle &+ (-1)^k \langle \phi, \tau'_k \rangle \langle \psi, \partial\tau''_{k+1} \rangle \\ &= \langle \delta\phi, \tau'_{k+1} \rangle \langle \psi, \tau''_l \rangle + (-1)^k \langle \phi, \tau'_k \rangle \langle \delta\psi, \tau''_{l+1} \rangle \\ &= \langle \delta\phi \times \psi, \tau \rangle + (-1)^k \langle \phi \times \delta\psi, \tau \rangle. \end{aligned}$$

Since τ is arbitrary, we have (10).

From this, (9) follows by setting $Y = X$ and writing $\phi \cup \psi = \Delta^{\#}(\phi \times \psi)$, where $\Delta\colon X \to X \times X$ denotes the diagonal map.