Attaching a 2-cell

Compare Proposition 1.26 in Hatcher.

Attach a 2-cell to X by means of the attaching map $f\colon S^1 \to X$ to form the space Y, so that we have the pushout square of spaces

$$\begin{array}{ccc} D^2 & \xrightarrow{g} & Y \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j} \\ S^1 & \xrightarrow{f} & X \end{array} \qquad (1)$$

We wish to know the effect on the fundamental group.

Choose a basepoint $d_1 \in S^1$ and put $x_1 = f(d_1)$.

THEOREM 2 *Given the above data, we have the pushout square of groups*

$$\begin{array}{ccc} \pi_1(D^2, d_1) & \xrightarrow{g_*} & \pi_1(Y, x_1) \\ \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle j_*} \\ \pi_1(S^1, d_1) & \xrightarrow{f_*} & \pi_1(X, x_1) \end{array}$$

Here, $\pi_1(D^2)$ is trivial because D^2 is convex and hence contractible, and $\pi_1(S^1) = \mathbb{Z}$, generated by $[\omega_1]$. Moreover, we don't wish to assume f is based; let x_0 be some other point of X, with h a path from x_0 to x_1. By combining with the isomorphisms β_h of Proposition 1.5, we deduce the more general version.

COROLLARY 3 *We have the pushout square of groups*

$$\begin{array}{ccc} \{1\} & \longrightarrow & \pi_1(Y, x_0) \\ \uparrow & & \uparrow{\scriptstyle j_*} \\ \mathbb{Z} & \longrightarrow & \pi_1(X, x_0) \end{array}$$

where the lower homomorphism takes $1 \in \mathbb{Z}$ *to* $\alpha = \beta_h(f_*[\omega_1])$. □

Interpretation By definition of pushouts, given any group H and homomorphisms $\phi_1\colon \{1\} \to H$ and $\phi_2\colon \pi_1(X, x_0) \to H$ that agree on $\mathbb{Z}$, there is a unique homomorphism $\phi\colon \pi_1(Y, x_0) \to H$ that makes the whole diagram commute. Thus ϕ_1 is trivial and $\phi_2(\alpha) = 1$. Then $\operatorname{Ker} \phi_2$ is a normal subgroup of $\pi_1(X, x_0)$ that contains α. Let N be the smallest normal subgroup of $\pi_1(X, x_0)$ that contains α (it is the intersection of all such subgroups). Then we may identify $\pi_1(Y, x_0)$ with the quotient group $\pi_1(X, x_0)/N$ and j_* with the natural quotient homomorphism.

Proof of Theorem The given pushout square diagram (1) does not lend itself to direct application of van Kampen's Theorem. The key idea is to *attach a collar*. Define the collar $V = \{x \in D^2 : \|x\| > 1/2\}$ of S^1 in D^2; it contains S^1 as a deformation retract. Now we form the expanded diagram of spaces, which contains diagram (1),

$$\begin{array}{ccccc}
e^2 & \xrightarrow{\subset} & D^2 & \xrightarrow{g} & Y \\
\uparrow{\scriptstyle\subset} & & \uparrow{\scriptstyle\subset} & & \uparrow{\scriptstyle\subset} \\
V - S^1 & \xrightarrow{\subset} & V & \longrightarrow & X \cup g(V) \\
 & & \uparrow{\scriptstyle\subset} & & \uparrow{\scriptstyle\subset} \\
 & & S^1 & \xrightarrow{f} & X
\end{array}$$

It contains *five* pushout squares, as pushout squares can be stacked.

Now we *can* apply van Kampen, taking $A_1 = e^2$ and $A_2 = X \cup g(V)$ and a basepoint $d_2 \in V - S^1$ that retracts to d_1, to obtain the pushout square of groups

$$\begin{array}{ccc}
\pi_1(e^2, d_2) & \xrightarrow{g_*} & \pi_1(Y, y_2) \\
\uparrow & & \uparrow{\scriptstyle j_*} \\
\pi_1(V - S^1, d_2) & \longrightarrow & \pi_1(X \cup g(V), y_2)
\end{array} \qquad (4)$$

where $y_2 = g(d_2)$.

This is not quite what we want. The basepoint y_2 is particularly inconvenient, as it does not lie in X. We may change any of the groups by an isomorphism. The retraction $V - S^1 \subset V \to S^1$ induces an isomorphism $\pi_1(V - S^1, d_2) \cong \pi_1(S^1, d_1)$. The induced retraction $s\colon X \cup g(V) \to X$ induces an isomorphism $s_*\colon \pi_1(X \cup g(V), y_2) \cong \pi_1(X, x_1)$, and the resulting homomorphism $\pi_1(S^1, d_1) \to \pi_1(X, x_1)$ is clearly f_*. Finally, we take k to be a path from x_1 to y_2 such that $s \circ k$ is the constant path at x_1 and use the commutative diagram

$$\begin{array}{ccc}
\pi_1(Y, y_2) & \xrightarrow{\beta_k} & \pi_1(Y, x_1) \\
\uparrow{\scriptstyle j_*} & & \uparrow{\scriptstyle j_*} \\
\pi_1(X \cup g(V), y_2) & \xrightarrow{\beta_k} & \pi_1(X \cup g(V), x_1) \\
\downarrow{\scriptstyle s_*} & \nearrow{\scriptstyle \cong} & \\
\pi_1(X, x_1) & &
\end{array}$$

to fix up the right side of diagram (4). $\square$