## Attaching a 2-cell

Compare Proposition 1.26 in Hatcher.

Attach a 2-cell to X by means of the attaching map  $f: S^1 \to X$  to form the space Y, so that we have the pushout square of spaces

$$D^{2} \xrightarrow{g} Y$$

$$\uparrow^{i} \qquad \uparrow^{j} \qquad (1)$$

$$S^{1} \xrightarrow{f} X$$

We wish to know the effect on the fundamental group.

Choose a basepoint  $d_1 \in S^1$  and put  $x_1 = f(d_1)$ .

THEOREM 2 Given the above data, we have the pushout square of groups

$$\pi_1(D^2, d_1) \xrightarrow{g_*} \pi_1(Y, x_1)$$

$$\uparrow^{i_*} \qquad \uparrow^{j_*}$$

$$\pi_1(S^1, d_1) \xrightarrow{f_*} \pi_1(X, x_1)$$

Here,  $\pi_1(D^2)$  is trivial because  $D^2$  is convex and hence contractible, and  $\pi_1(S^1) = \mathbb{Z}$ , generated by  $[\omega_1]$ . Moreover, we don't wish to assume f is based; let  $x_0$  be some other point of X, with h a path from  $x_0$  to  $x_1$ . By combining with the isomorphisms  $\beta_h$  of Proposition 1.5, we deduce the more general version.

COROLLARY 3 We have the pushout square of groups

$$\{1\} \longrightarrow \pi_1(Y, x_0)$$

$$\downarrow \qquad \qquad \downarrow^{j_*}$$

$$\mathbb{Z} \longrightarrow \pi_1(X, x_0)$$

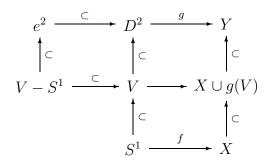
where the lower homomorphism takes  $1 \in \mathbb{Z}$  to  $\alpha = \beta_h(f_*[\omega_1])$ .

Interpretation By definition of pushouts, given any group H and homomorphisms  $\phi_1: \{1\} \to H$  and  $\phi_2: \pi_1(X, x_0) \to H$  that agree on  $\mathbb{Z}$ , there is a unique homomorphism  $\phi: \pi_1(Y, x_0) \to H$  that makes the whole diagram commute. Thus  $\phi_1$  is trivial and  $\phi_2(\alpha) = 1$ . Then Ker  $\phi_2$  is a normal subgroup of  $\pi_1(X, x_0)$  that contains  $\alpha$ . Let N be the smallest normal subgroup of  $\pi_1(X, x_0)$  that contains  $\alpha$  (it is the intersection of all such subgroups). Then we may identify  $\pi_1(Y, x_0)$  with the quotient group  $\pi_1(X, x_0)/N$  and  $j_*$  with the natural quotient homomorphism.

*Proof of Theorem* The given pushout square diagram (1) does not lend itself to direct application of van Kampen's Theorem. The key idea is to *attach a collar*. Define the

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collar  $V = \{x \in D^2 : ||x|| > 1/2\}$  of  $S^1$  in  $D^2$ ; it contains  $S^1$  as a deformation retract. Now we form the expanded diagram of spaces, which contains diagram (1),

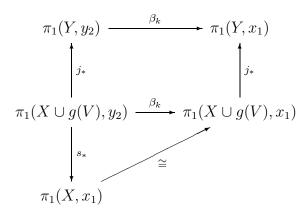


It contains *five* pushout squares, as pushout squares can be stacked.

Now we can apply van Kampen, taking  $A_1 = e^2$  and  $A_2 = X \cup g(V)$  and a basepoint  $d_2 \in V - S^1$  that retracts to  $d_1$ , to obtain the pushout square of groups

where  $y_2 = g(d_2)$ .

This is not quite what we want. The basepoint  $y_2$  is particularly inconvenient, as it does not lie in X. We may change any of the groups by an isomorphism. The retraction  $V - S^1 \subset V \to S^1$  induces an isomorphism  $\pi_1(V - S^1, d_2) \cong \pi_1(S^1, d_1)$ . The induced retraction  $s: X \cup g(V) \to X$  induces an isomorphism  $s_*: \pi_1(X \cup g(V), y_2) \cong$  $\pi_1(X, x_1)$ , and the resulting homomorphism  $\pi_1(S^1, d_1) \to \pi_1(X, x_1)$  is clearly  $f_*$ . Finally, we take k to be a path from  $x_1$  to  $y_2$  such that  $s \circ k$  is the constant path at  $x_1$  and use the commutative diagram



to fix up the right side of diagram (4).  $\Box$ 

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