Barycentric Subdivision

This note parallels material on pages 119–123 of Hatcher.

Linear simplices Let $Y$ be a convex subspace of $\mathbb{R}^q$. Given points $v_0, v_1, \ldots, v_n$ in $Y$ (not necessarily independent), recall the linear $n$-simplex

$$\lambda = [v_0, v_1, \ldots, v_n]_n: \Delta^n \rightarrow Y,$$

which is the restriction to $\Delta^n$ of the linear map $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^q$ that sends the standard basis vector $e_i$ to $v_i$ for each $i$. The barycenter or centroid of this linear simplex is the point $b(\lambda) = \sum_i v_i/(n+1)$. The linear simplices generate the group $LC_n(Y)$ of linear chains on $Y$.

Cone operator Given a point $y \in Y$, we define the cone operator homomorphism $C_y: LC_n(Y) \rightarrow LC_{n+1}(Y)$ by

$$C_y \lambda = C_y[v_0, v_1, \ldots, v_n] = [y, v_0, v_1, \ldots, v_n].$$

Geometrically, we join everything to the point $y$.

We compute

$$\partial C_y \lambda = [v_0, v_1, \ldots, v_n] - \sum_{i=0}^n [y, v_0, \ldots, \hat{v}_i, \ldots, v_n] = \lambda - C_y \partial \lambda,$$

assuming that $n > 0$. If $n = 0$, this calculation is not valid (unless one introduces the empty (-1)-simplex $[]$ as in Hatcher); instead we find $\partial C_y[v_0] = \partial [y, v_0] = [v_0] - [y]$. We combine these as

$$\partial C_y \lambda + C_y \partial \lambda = \lambda - r_y \lambda \quad \text{in} \quad LC_n(Y)$$

for all $n$, where the chain map $r_y: LC(Y) \rightarrow LC(Y)$ is given by $r_y \lambda = 0$ for $n > 0$ and $r_y[v_0] = [y]$. Thus $C_y$ is a contracting chain homotopy for $LC(Y)$, which expresses algebraically the contraction of the convex subspace $Y$ to the point $y$.

Barycentric subdivision We define the barycentric subdivision first on linear simplices, to produce a chain map $S_n: LC_n(Y) \rightarrow LC_{n}(Y)$. Geometrically, we proceed by induction; once the faces of $\lambda$ have been subdivided, we join everything to the barycenter $b(\lambda)$ of $\lambda$.

Algebraically, we begin the induction with $S_0 = 1$, and continue with

$$S_n \lambda = C_{b(\lambda)} S_{n-1} \partial \lambda \quad \text{for} \quad n > 0.$$  

(Of course, $S_n = 0$ for $n < 0$.) The form of this definition, with equation (3), implies that $S$ is a chain map. For $n \geq 2$ we compute

$$\partial S_n \lambda = \partial C_{b(\lambda)} S_{n-1} \partial \lambda = S_{n-1} \partial \lambda - C_{b(\lambda)} \partial S_{n-1} \partial \lambda = S_{n-1} \partial \lambda,$$

since by induction $\partial S_{n-1} \partial \lambda = S_{n-2} \partial \partial \lambda = 0$. For $n = 1$, there is an extra term $r_y S_0 \partial \lambda = r_y \partial \lambda$, which vanishes.

Chain homotopy We need a chain homotopy $T$ between $S$ and the identity chain map $1$, i.e. $T_n: LC_n(Y) \rightarrow LC_{n+1}(Y)$ that satisfies

$$\partial \circ T_n + T_{n-1} \circ \partial = 1 - S_n \quad \text{for all} \quad n.$$  

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Geometrically (see the picture on page 122), we subdivide the face $\Delta^n \times 0$ of the prism $\Delta^n \times I$, leave the face $\Delta^n \times 1$ alone, and join the barycenter $(b(\lambda), 0)$ to the already subdivided faces $\partial \Delta^n \times I$ of the prism.

Algebraically, we begin the induction with $T_0[v_0] = 0$. (Of course, $T_n = 0$ for $n < 0$. Hatcher uses in effect $T_0[v_0] = [v_0, v_0]$, but both choices satisfy equation (5) for $n = 0$.) We continue with

$$T_n \lambda = C_{b(\lambda)}(\lambda - T_{n-1} \partial \lambda) \quad \text{for } n > 0. \quad (6)$$

We verify equation (5) for $n > 0$ by using equation (3),

$$\partial T_n \lambda = \partial C_{b(\lambda)}(\lambda - T_{n-1} \partial \lambda) = \lambda - T_{n-1} \partial \lambda - C_{b(\lambda)} \partial \lambda + C_{b(\lambda)} \partial T_{n-1} \partial \lambda.$$ 

By induction, we have, from equation (5) for $n - 1$,

$$C_{b(\lambda)} \partial T_{n-1} \partial \lambda + C_{b(\lambda)} T_n \partial \partial \lambda = C_{b(\lambda)} \partial \lambda - C_{b(\lambda)} S_{n-1} \partial \lambda.$$ 

The second term on the left vanishes, and the second term on the right is $S_n \lambda$, by definition.

The following property of $S_n$ and $T_n$ is immediate.

**Lemma 7** Let $A: \mathbb{R}^q \to \mathbb{R}^r$ be a linear map, and $Y'$ a convex subspace of $\mathbb{R}^r$ such that $A(Y) \subset Y'$. Then the chain map $A\#: LC_n(Y) \to LC_n(Y')$ commutes with $S_n$ and $T_n$, $S_n \circ A\# = A\# \circ S_n$ and $T_n \circ A\# = A\# \circ T_n$. \(\square\)

**General singular simplices** We extend the definition of $S_n$ and $T_n$ to a singular $n$-simplex $\sigma: \Delta^n \to X$ of any space $X$ by using the chain map $\sigma\#: LC_n(\Delta^n) \subset C_n(\Delta^n) \to C_n(X)$ and noting that $\sigma = \sigma\# [e_0, e_1, \ldots, e_n]$. For all $n$, we define

$$S_n \sigma = \sigma\# [e_0, e_1, \ldots, e_n] \quad (8)$$

and

$$T_n \sigma = \sigma\# [e_0, e_1, \ldots, e_n]. \quad (9)$$

These are consistent with previous definitions by Lemma 7 if $X$ happens to be a real vector space and $\sigma$ is a linear map.

To verify that $S$ remains a chain map, we compute

$$\partial S_n \sigma = \partial \sigma\# S_n[e_0, e_1, \ldots, e_n] = \sigma\# S_{n-1} \partial [e_0, e_1, \ldots, e_n],$$

since $\sigma\#$ and the linear version of $S$ are chain maps. On the other side, equation (8) for the face $d_i \sigma = \sigma \circ \eta_i$ gives

$$S_{n-1} d_i \sigma = \sigma\# \eta_i \# S_{n-1} [e_0, e_1, \ldots, e_{n-1}].$$

By Lemma 7, we may rewrite this as

$$\sigma\# S_{n-1} \eta_i \# [e_0, e_1, \ldots, e_{n-1}] = \sigma\# S_{n-1} [e_0, \ldots, \hat{e}_i, \ldots, e_n].$$

Now we take alternating sums over $i$.

A similar proof shows that $T$ continues to satisfy equation (5).