The Eilenberg–Zilber Theorem

The chain complexes $C(X\times Y)$ and $C(X)\otimes C(Y)$ Throughout, we write $C(X)$ for the whole singular chain complex of the space X, with the abelian group $C_n(X)$ in degree n. There are geometric reasons, at least for nice spaces, for thinking that these two chain complexes should be related somehow. The Eilenberg–Zilber Theorem provides the concrete answer.

However, these chain complexes do not at all resemble each other. The abelian group $C_n(X\times Y)$ is freely generated by the singular n-simplices $\sigma\colon \Delta^n \to X\times Y$, where Δ^n denotes the standard n-simplex. The map σ has two coordinates, $\sigma_1\colon \Delta^n \to X$ and $\sigma_2\colon \Delta^n \to Y$, and we find it convenient to write $\sigma = (\sigma_1, \sigma_2)$.

The component $(C(X)\otimes C(Y))_n = \bigoplus_{k+l=n} C_k(X)\otimes C_l(Y)$ in degree n of $C(X)\otimes C(Y)$ is the free abelian group generated by the elements $\sigma\otimes\tau$, where $\sigma\colon \Delta^k \to X$ and $\tau\colon \Delta^l \to Y$ are singular simplices of X and Y with $k+l=n$.

Both $C(X\times Y)$ and $C(X)\otimes C(Y)$ are *augmented* chain complexes; the second has the augmentation $\epsilon\otimes\epsilon\colon C_0(X)\otimes C_0(Y) \to \mathbb{Z}\otimes\mathbb{Z} \cong \mathbb{Z}$.

THEOREM 1 *For any spaces X and Y:*

(a) *There exists a natural transformation $\alpha\colon C(-\times -) \to C(-)\otimes C(-)$ of augmented chain complexes, consisting of augmented chain maps*

$$\alpha(X,Y)\colon C(X\times Y) \longrightarrow C(X)\otimes C(Y),$$

and it is unique up to natural chain homotopy.

(b) *There exists a natural transformation $\beta\colon C(-)\otimes C(-) \to C(-\times -)$ of augmented chain complexes, consisting of augmented chain maps*

$$\beta(X,Y)\colon C(X)\otimes C(Y) \longrightarrow C(X\times Y),$$

amd it is unique up to natural chain homotopy.

(c) *The chain maps $\alpha(X,Y)$ and $\beta(X,Y)$ are naturally homotopy inverse chain homotopy equivalences.*

The proof is entirely categorical, with essentially no geometry, and is known as the method of *acyclic models*. It imitates Lemma 3.1 in Hatcher, which constructs a chain map $E\to G$ between chain complexes, where E has a freeness property and G has an exactness property.

Acyclicity

LEMMA 2 *For any k and l, the augmented chain complexes $C(\Delta^k\times\Delta^l)$ and $C(\Delta^k)\otimes C(\Delta^l)$ are acyclic. Explicitly, the sequences*

$$\dots \xrightarrow{\partial} C_2(\Delta^k\times\Delta^l) \xrightarrow{\partial} C_1(\Delta^k\times\Delta^l) \xrightarrow{\partial} C_0(\Delta^k\times\Delta^l) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \qquad (3)$$

and

$$\dots \xrightarrow{\partial} (C(\Delta^k)\otimes C(\Delta^l))_1 \xrightarrow{\partial} (C(\Delta^k)\otimes C(\Delta^l))_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \qquad (4)$$

are exact.

Proof This is immediate for (3), as the space $\Delta^k \times \Delta^l$ is contractible.

For (4), we need slightly more detail. Because Δ^k is contractible, $C(\Delta^k)$ is chain homotopy equivalent to $C(P)$, where P denotes a one-point space. Recall that for all $n \geq 0$, $C_n(P) \cong \mathbb{Z}$, generated by $\sigma_n\colon \Delta^n \to P$, with the boundary operator given by $\partial\sigma_{2n} = \sigma_{2n-1}$ and $\partial\sigma_{2n-1} = 0$ for $n > 0$. Denote by $C'(P)$ the subcomplex of $C(P)$ with $C'_0(P) = C_0(P)$ and $C'_n(P) = 0$ for all $n \neq 0$, included by $i\colon C'(P) \subset C(P)$. We claim that $C'(P)$ is a chain deformation retract of $C(P)$; homologically, σ_{2n} and σ_{2n-1} cancel each other out for all $n > 0$. The retraction chain map $r\colon C(P) \to C'(P)$ is obvious (and unique). Explicitly, the needed chain homotopy s is given on generators by $s\sigma_0 = 0$ and, for $n > 0$, $s\sigma_{2n-1} = \sigma_{2n}$ and $s\sigma_{2n} = 0$. [In detail, for $n > 0$, $\partial s\sigma_{2n} + s\partial\sigma_{2n} = 0 + s\sigma_{2n-1} = \sigma_{2n} = \sigma_{2n} - ir\sigma_{2n}$ and $\partial s\sigma_{2n-1} + s\partial\sigma_{2n-1} = \partial\sigma_{2n} + 0 = \sigma_{2n-1} = \sigma_{2n-1} - ir\sigma_{2n-1}$. Also, $\partial s\sigma_0 = 0 = \sigma_0 - ir\sigma_0$.] Thus $C'(P)$ is chain homotopy equivalent to $C(P)$ and hence to $C(\Delta^k)$ for all k. It follows that $C(\Delta^k) \otimes C(\Delta^l)$ is chain homotopy equivalent to $C'(P) \otimes C'(P)$, which is obviously acyclic. □

Freeness We may regard the identity map $\mathrm{id}\colon \Delta^n \to \Delta^n$ as an n-chain $\delta_n \in C_n(\Delta^n)$. We write *Top* for the category of topological spaces and *Ab* for the category of abelian groups.

LEMMA 5 *Let K: Top $\to$ Ab be any functor.*

(a) *A natural transformation $\theta\colon C_n(-) \to K$, consisting of homomorphisms $\theta X\colon C_n(X) \to KX$, is completely determined by the value $z = (\theta\Delta^n)\delta_n \in K\Delta^n$;*

(b) *Given any element $z \in K\Delta^n$, there exists a natural transformation θ with $(\theta\Delta^n)\delta_n = z$.*

Proof The group $C_n(X)$ is generated by the singular simplices $\sigma\colon \Delta^n \to X$. Thus the homomorphism θX is determined by the values $(\theta X)\sigma$. We claim that these are given, for any space X, by the formula

$$(\theta X)\sigma = (K\sigma)z \quad \text{in } KX, \tag{6}$$

which proves (a). To see this, we rewrite $\sigma = \sigma \circ \mathrm{id}$ as $\sigma = \sigma_\#\delta_n$ and use naturality with respect to the map σ to write

$$(\theta X)\sigma = (\theta X)\sigma_\#\delta_n = (K\sigma)(\theta\Delta^n)\delta_n.$$

For (b), given any element $z \in K\Delta^n$, we *define* θX on each σ by (6) and extend it uniquely to a homomorphism $\theta X\colon C_n(X) \to KX$. To see that θ so defined is natural, we take any map $f\colon X \to Y$ and check that $Kf \circ \theta X = \theta Y \circ f_\#$. We evaluate each side on any generator $\sigma \in C_n(X)$ and compare $(Kf)(\theta X)\sigma = (Kf)(K\sigma)z$ with $(\theta Y)f_\#\sigma = (\theta Y)(f \circ \sigma) = K(f \circ \sigma)z$. These agree since K is a functor. □

This is not the lemma we want. We need two more elaborate versions, with analogous proofs. We regard the diagonal map $(\mathrm{id}, \mathrm{id})\colon \Delta^n \to \Delta^n \times \Delta^n$ as an n-chain $d_n \in C_n(\Delta^n \times \Delta^n)$.

LEMMA 7 *Let $F(X,Y) = C_n(X \times Y)$ and let K: Top $\times$ Top $\to$ Ab be any functor.*

(a) *A natural transformation $\theta\colon F \to K$ is completely determined by the value $z = \theta(\Delta^n, \Delta^n)d_n \in K(\Delta^n, \Delta^n)$.*

(b) *Given any element* $z \in K(\Delta^n, \Delta^n)$, *there exists a natural transformation* θ *with* $\theta(\Delta^n, \Delta^n)d_n = z$.

Proof This time, θ is given in terms of z by the formula

$$\theta(X, Y)\sigma = K(\sigma_1, \sigma_2)z, \tag{8}$$

where $\sigma\colon \Delta^n \to X \times Y$ is a singular simplex with coordinates $\sigma_1\colon \Delta^n \to X$ and $\sigma_2\colon \Delta^n \to Y$. We write $\sigma = \sigma_\# \delta_n = (\sigma_1, \sigma_2)_\# \delta_n = (\sigma_1 \times \sigma_2)_\# d_n$; then by naturality,

$$\theta(X, Y)\sigma = \theta(X, Y)(\sigma_1 \times \sigma_2)_\# d_n = K(\sigma_1, \sigma_2)\theta(\Delta^n, \Delta^n)d_n = K(\sigma_1, \sigma_2)z.$$

Given any $z \in K(\Delta^n, \Delta^n)$, we define the homomorphism $\theta(X, Y)$ by (8). For any maps $f\colon X \to X'$ and $g\colon Y \to Y'$, we check that $K(f, g)\circ\theta(X, Y) = \theta(X', Y')\circ(f\times g)_\#$, much as in the proof of Lemma 5. □

A similar result holds for the chain complex $C(X) \otimes C(Y)$.

Lemma 9 *Let* $F(X, Y) = C_k(X)\otimes C_l(X)$ *and let* $K\colon$ *Top* $\times$ *Top* $\to$ *Ab be any functor. Then:*

(a) *A natural transformation* $\theta\colon F \to K$ *is completely determined by the value* $z = \theta(\Delta^k, \Delta^l)(\delta_k \otimes \delta_l) \in K(\Delta^k, \Delta^l)$.

(b) *Given any element* $z \in K(\Delta^k, \Delta^l)$, *there exists a natural transformation* θ *with* $\theta(\Delta^k, \Delta^l)(\delta_k\otimes\delta_l) = z$.

Proof This time, θ is given by the formula

$$\theta(X, Y)(\sigma \otimes \tau) = K(\sigma, \tau)z, \tag{10}$$

where $\sigma\colon \Delta^k \to X$ and $\tau\colon \Delta^l \to Y$ are singular simplices. If we choose z and define θ by (10), to show that θ so defined is natural, we need to check that, for any maps $f\colon X \to X'$ and $g\colon Y \to Y'$, we have $K(f, g) \circ \theta(X, Y) = \theta(X', Y') \circ (f_\#\otimes g_\#)$. □

We combine these two lemmas with acyclicity to deduce the fundamental property we need. Consider the diagram of functors *Top* $\times$ *Top* $\to$ *Ab* and natural transformations

$$\begin{array}{ccccc} & & F(X,Y) & & \\ & \swarrow^{\widetilde{\theta}(X,Y)} & \downarrow{\scriptstyle \theta(X,Y)} & \searrow^{0} & \\ K(X,Y) & \xrightarrow{\partial} & K'(X,Y) & \xrightarrow{\partial} & K''(X,Y) \end{array} \tag{11}$$

Lemma 12 *Suppose that* $F\colon$ *Top* $\times$ *Top* $\to$ *Ab is any of the functors* (i) $F(X, Y) = C_n(X\times Y)$ *(for a fixed* n*),* (ii) $F(X, Y) = C_k(X) \otimes C_l(Y)$ *(for fixed* k *and* l*), or* (iii) $F(X, Y) = (C(X)\otimes C(Y))_n$ *(for a fixed* n*), and that the bottom row of diagram* (11) *is exact for* $X = \Delta^r$ *and* $Y = \Delta^s$ *and all* r *and* s*. If the natural transformation* θ *satisfies* $\partial \circ \theta = 0$, *there exists a natural transformation* $\widetilde{\theta}$ *that lifts* θ, *i. e.* $\partial \circ \widetilde{\theta} = \theta$.

Proof In Case (i), Lemma 7 shows that θ is determined by the value $z \in K'(\Delta^n, \Delta^n)$ and that $\widetilde{\theta}$ may be defined by choosing an element $\widetilde{z} \in K(\Delta^n, \Delta^n)$. It also shows that $\partial \circ \widetilde{\theta} = \theta$ will hold provided that $\partial\widetilde{z} = z$, which can be arranged in view of the assumed exactness.

Similarly, in Case (ii), Lemma 9 shows that θ is determined by $z \in K'(\Delta^k, \Delta^l)$ and that $\widetilde{\theta}$ may be constructed by choosing $\widetilde{z} \in K(\Delta^k, \Delta^l)$. We again need $\widetilde{z}$ to satisfy $\partial\widetilde{z} = z$, which again is possible by exactness.

Since $(C(X)\otimes C(Y))_n = \bigoplus_{k+l=n} C_k(X) \otimes C_l(Y)$, Case (iii) follows immediately from Case (ii). □

Constructing chain maps With the help of Lemma 12, we can now proceed, formally identically to the statement and proof of Lemma 3.1 in Hatcher.

Lemma 13 *Suppose that each of E and G is either of the functorial augmented chain complexes given on (X,Y) as $C(X\times Y)$ or $C(X) \otimes C(Y)$. Then there is a natural chain map $\theta(X,Y)\colon E(X,Y) \to G(X,Y)$, and it is unique up to natural chain homotopy.*

Proof We construct the natural transformation $\theta_n(X,Y)\colon E_n(X,Y) \to G_n(X,Y)$, a component of θ, by induction on n, where $E_n(X,Y)$ and $G_n(X,Y)$ denote the components of $E(X,Y)$ and $G(X,Y)$ in degree n. We begin with the general induction step. Given $n \geq 2$, suppose we have θ_{n-1} and θ_{n-2} that make the right square in the diagram

$$\begin{array}{ccccc}
E_n(X,Y) & \xrightarrow{\partial} & E_{n-1}(X,Y) & \xrightarrow{\partial} & E_{n-2}(X,Y) \\
\downarrow{\scriptstyle \theta_n(X,Y)} & & \downarrow{\scriptstyle \theta_{n-1}(X,Y)} & & \downarrow{\scriptstyle \theta_{n-2}(X,Y)} \\
G_n(X,Y) & \xrightarrow{\partial} & G_{n-1}(X,Y) & \xrightarrow{\partial} & G_{n-2}(X,Y)
\end{array}$$

commute. We fill in θ_n to make the left square commute, by applying Lemma 12 with $F = E_n$, $K = G_n$, $K' = G_{n-1}$ and $K'' = G_{n-2}$; the exactness hypothesis holds by Lemma 2.

For the first two steps in the induction, namely $n = 0$ and $n = 1$, we use the slightly modified diagram

$$\begin{array}{ccccccc}
E_1(X,Y) & \xrightarrow{\partial} & E_0(X,Y) & \xrightarrow{\epsilon} & \mathbb{Z} & & \\
\downarrow{\scriptstyle \theta_1(X,Y)} & & \downarrow{\scriptstyle \theta_0(X,Y)} & & \downarrow{\scriptstyle =} & & \\
G_1(X,Y) & \xrightarrow{\partial} & G_0(X,Y) & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

[Here, we regard $\mathbb{Z}$ as the constant functor of (X,Y).] Again, we apply Lemma 12 to fill in θ_0 first, so that θ_0 preserves the augmentation ϵ, then θ_1.

Now suppose that θ' is another natural augmented chain map, with components θ'_n. We need a chain homotopy s with component natural transformations $s_n(X,Y)\colon E_n(X,Y) \to G_{n+1}(X,Y)$ that satisfy

$$\partial \circ s_n + s_{n-1} \circ \partial = \theta'_n - \theta_n\colon E_n \longrightarrow G_n \tag{14}$$

for all n, where $s_n = 0$ vacuously for $n < 0$. We first assume that $n \geq 2$ and that we have s_{n-1} that satisfies (14) (with $n-1$ in place of n). We wish to fill in s_n in the diagram

$$\begin{array}{ccccc}
 & & E_n(X,Y) & \xrightarrow{\partial} & E_{n-1}(X,Y) \\
 & \swarrow{\scriptstyle s_n} & \downarrow{\scriptstyle \theta_n}\ \downarrow{\scriptstyle \theta'_n} & \swarrow{\scriptstyle s_{n-1}} & \downarrow{\scriptstyle \theta_{n-1}}\ \downarrow{\scriptstyle \theta'_{n-1}} \\
G_{n+1}(X,Y) & \xrightarrow{\partial} & G_n(X,Y) & \xrightarrow{\partial} & G_{n-1}(X,Y)
\end{array}$$

to satisfy (14). We may apply Lemma 12 to the natural transformation $\theta'_n - \theta_n - s_{n-1} \circ \partial \colon E_n \to G_n$, since

$$\begin{aligned}\partial \circ (\theta'_n - \theta_n - s_{n-1} \circ \partial) &= \partial \circ \theta'_n - \partial \circ \theta_n - (\partial \circ s_{n-1}) \circ \partial \\ &= \theta'_{n-1} \circ \partial - \theta_{n-1} \circ \partial - (\theta'_{n-1} - \theta_{n-1} - s_{n-2} \circ \partial) \circ \partial \\ &= s_{n-2} \circ \partial \circ \partial = 0,\end{aligned}$$

taking $F = E_n$, $K = G_{n+1}$ etc, to produce s_n as required.

To begin the induction, we again modify the diagram to

$$\begin{array}{ccccccc} & & E_1(X,Y) & \xrightarrow{\partial} & E_0(X,Y) & \xrightarrow{\epsilon} & \mathbb{Z} \\ & \swarrow^{s_1} & \theta_1 \downarrow \downarrow \theta'_1 & \swarrow^{s_0} & \theta_0 \downarrow \downarrow \theta'_0 & & \downarrow = \\ G_2(X,Y) & \xrightarrow{\partial} & G_1(X,Y) & \xrightarrow{\partial} & G_0(X,Y) & \xrightarrow{\epsilon} & \mathbb{Z} \end{array}$$

To find s_0 satisfying $\partial \circ s_0 = \theta'_0 - \theta_0$, we check that $\epsilon \circ (\theta'_0 - \theta_0) = \epsilon - \epsilon = 0$. Then for s_1, we need

$$\partial \circ (\theta'_1 - \theta_1 - s_0 \circ \partial) = \theta'_0 \circ \partial - \theta_0 \circ \partial - \partial \circ s_0 \circ \partial = 0. \quad \square$$

Proof of the Theorem We apply Lemma 13 four times.

Proof of Theorem 1 For (a), we apply Lemma 13 with $E(X,Y) = C(X \times Y)$ and $G(X,Y) = C(X) \otimes C(Y)$ to obtain the natural chain map $\alpha(X,Y) \colon C(X \times Y) \to C(X) \otimes C(Y)$, unique up to natural chain homotopy.

For (b), we apply Lemma 13 with $E(X,Y) = C(X) \otimes C(Y)$ and $G(X,Y) = C(X \times Y)$ to obtain $\beta(X,Y) \colon C(X) \otimes C(Y) \to C(X \times Y)$, unique up to natural chain homotopy.

For (c), we apply Lemma 13 with $E(X,Y) = G(X,Y) = C(X \times Y)$ to the two natural chain maps $\beta(X,Y) \circ \alpha(X,Y), \mathrm{id} \colon E(X,Y) \to G(X,Y)$ to deduce that they are chain homotopic, making β is a left homotopy inverse to α. Finally, we apply Lemma 13 with $E(X,Y) = G(X,Y) = C(X) \otimes C(Y)$ to see that $\alpha(X,Y) \circ \beta(X,Y)$ is chain homotopic to the identity chain map id. $\square$

Remark The Alexander–Whitney formula gives a simple explicit candidate for α. Given a singular n-simplex $\sigma \colon \Delta^n \to X \times Y$, we may take

$$\alpha(X,Y)\sigma = \sum_{k=0}^{n} (\sigma_1 \circ \lambda^{(n-k)}) \otimes (\sigma_2 \circ \rho^{(k)}), \tag{15}$$

where $\lambda^{(n-k)} \colon \Delta^k \to \Delta^n$ is the linear map given on vertices by $\lambda^{(n-k)} e_i = e_i$ for $0 \le i \le k$, and $\rho^{(k)} \colon \Delta^{n-k} \to \Delta^n$ is given by $\rho^{(k)} e_j = e_{k+j}$ for $0 \le j \le n-k$. [It does not seem to have a direct geometric interpretation.]

On the other hand, the formula for β (which we do not give, or need) is far more complicated (see Hatcher, p. 277–278), but can be interpreted in terms of cutting up the product $\Delta^j \times \Delta^k$ into $(j+k)$-simplices. It directly induces the *homology cross product*, which is well defined as $\beta(X,Y)$ is unique up to chain homotopy:

$$\begin{aligned}\times \colon H_k(X) \times H_l(Y) = H_k(C(X)) \times H_l(C(Y)) &\xrightarrow{\otimes} H_{k+l}(C(X) \otimes C(Y)) \\ &\xrightarrow{\beta(X,Y)_*} H_{k+l}(C(X \times Y)) = H_{k+l}(X \times Y).\end{aligned}$$