The Homotopy Extension Property

This note augments material in Hatcher, Chapter 0.

We assume throughout that A is a *closed* subspace of X.

The Homotopy Extension Property Not all inclusions $A \subset X$ are created equal. One might expect that if A is contractible, the quotient map $q\colon X \to X/A$ should be a homotopy equivalence. This is not always true. We need a definition.

DEFINITION 1 The pair (X, A) *has the homotopy extension property* (HEP) if given any space Y, a homotopy $f_t\colon A \to Y$, and a map $g_0\colon X \to Y$ such that $f_0 = g_0|A$, there exists a homotopy $g_t\colon X \to Y$ that starts from the given map g_0 and extends the homotopy f_t, in the sense that $f_t = g_t|A$ for all t.

We are interested in knowing that such a homotopy exists; we are *not* interested in seeing one written out explicitly. We now have a theorem, which is a restatement of Proposition 0.17 in Hatcher. We do not repeat the proof here, as we have nothing to add. Note that A/A is just a complicated way of writing the basepoint in X/A.

THEOREM 2 *If the pair (X, A) has the* HEP *and A is contractible, then the quotient map $q\colon (X, A) \to (X/A, A/A)$ is a homotopy equivalence of pairs.* □

LEMMA 3 *The pair (X, A) has the* HEP *if and only if $A \times I \cup X \times 0$ is a retract of $X \times I$.*

Proof Suppose $r\colon X \times I \to A \times I \cup X \times 0$ is a retraction. A homotopy f_t and map g_0 as above combine to give a map $k\colon A \times I \cup X \times 0 \to Y$. The desired homotopy is $g_t(x) = k(r(x, t))$.

Conversely, take $Y = A \times I \cup X \times 0$, $f_t(a) = (a, t)$, and $g_0(x) = (x, 0)$. If (X, A) has the HEP, the resulting homotopy $g_t\colon X \to Y$ can be rewritten as a retraction $X \times I \to Y$. □

COROLLARY 4 *If (X, A) has the* HEP, *so does $(W \times X, W \times A)$ for any space W.*

Proof If $r\colon X \times I \to A \times I \cup X \times 0$ is a retraction, so is

$$\mathrm{id}_W \times r\colon W \times X \times I \to W \times A \times I \cup W \times X \times 0. \quad \square$$

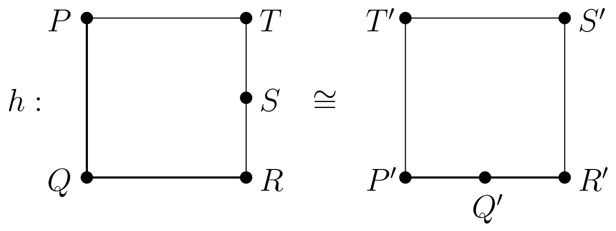
PROPOSITION 5 *If (X, A) has the* HEP, *so does the pair $(X \times I, A \times I \cup X \times 0)$.*

Proof By Lemma 3, we have to show that $(A \times I \cup X \times 0) \times I \cup (X \times I) \times 0$ is a retract of $(X \times I) \times I$. We can rewrite the subspace as $A \times (I \times I) \cup X \times (0 \times I \cup I \times 0)$. We claim there is a homeomorphism of pairs

$$h\colon (I \times I, 0 \times I \cup I \times 0) \cong (I \times I, I \times 0).$$

This homeomorphism converts the subspace of $X \times I \times I$ into $A \times I \times I \cup X \times I \times 0$, which is a retract of $X \times I \times I$ by Lemma 3, because $(X \times I, A \times I)$ has the HEP by Corollary 4.

To see that such a homeomorphism exists, we may treat $I \times I$ as the cone $C(\partial I^2)$ on the boundary square. There is an obvious homeomorphism $h\colon \partial I^2 \to \partial I^2$ (which we refrain from writing out explicitly) that satisfies $h(P) = P'$, $h(Q) = Q'$, etc.



□

With obvious modifications, we can also prove the following.

PROPOSITION 6 *If (X, A) has the* HEP, *so does the pair $(X \times I, A \times I \cup X \times \partial I)$.* □

Deformation retracts The following substantial result is exactly Corollary 0.20 in Hatcher, but we rearrange the proof somewhat.

THEOREM 7 *If the pair (X, A) has the* HEP *and the inclusion $i: A \to X$ is a homotopy equivalence, then A is a deformation retract of X.*

Remark There are (at least) three different variations in the literature on the concept of deformation retract, not always consistently named. We follow Hatcher in requiring a retraction $r: X \to A$ that satisfies $\mathrm{id}_X \simeq i \circ r$ rel A. This is the strongest and most useful variation, and the only one we use.

Proof Let $g: X \to A$ be a homotopy inverse. A homotopy $g|A = g \circ i \simeq \mathrm{id}_A: A \to A$ extends by the HEP to a homotopy $g \simeq r: X \to A$; since $r|A = \mathrm{id}_A$, r is a retraction and is still a homotopy inverse to i.

Denote by J the interval $[-1, 1] \subset \mathbb{R}$. Let $f_t: i \circ r \simeq \mathrm{id}_X: X \to X$ be a homotopy. Define a homotopy $g_u: A \times J \to X$ by $g_u(a, t) = f_{\max(|t|,u)}(a)$, so that $g_u(a, t) = a$ whenever $t = \pm 1$ or $u = 1$. Also, define a map $h_0: X \times J \to X$ by

$$h_0(x,t) = \begin{cases} f_{|t|}(x) & \text{if } -1 \le t \le 0; \\ f_t(i(r(x))) & \text{if } 0 \le t \le 1. \end{cases}$$

Note that these agree at $t = 0$, since $i \circ r = i \circ r \circ i \circ r$. Since $(X \times J, A \times J)$ has the HEP by Corollary 4, the homotopy g_u extends to a homotopy $h_u: X \times J \to X$, i. e. a map $H: X \times J \times I \to X$. The map h_0 may be interpreted as a homotopy $\mathrm{id}_X \simeq i \circ r$. The other three sides of the rectangle $J \times I$ yield the desired homotopy $\mathrm{id}_X \simeq i \circ r$ rel A. (The picture is on page 17.) □

PROPOSITION 8 *If (X, A) has the* HEP, *then $A \times I \cup X \times 0$ is a deformation retract of $X \times I$.*

Proof The space $X \times 0$ is an obvious deformation retract of both $X \times I$ and $A \times I \cup X \times 0$. Proposition 5 allows us to apply Theorem 7. □

Remark There is an alternate direct (but less conceptual) proof. By Lemma 3, there is a retraction $r: X \times I \to A \times I \cup X \times 0$, which we write in coordinates as $r(x,t) = (r_1(x,t), r_2(x,t))$. The fact that r is a retraction is expressed by the equations

$$r_1(a,t) = a, \quad r_2(a,t) = t, \quad r_1(x,0) = x, \quad r_2(x,0) = 0, \tag{9}$$

where $a \in A$ and $x \in X$. We define the deformation retraction $s_u\colon X \times I \to X \times I$ by

$$s_u(x,t) = (r_1(x,ut), (1-u)t + ur_2(x,t)).$$

We can check by (9) that $s_0(x,t) = (x,t)$, $s_u(a,t) = (a,t)$, $s_u(x,0) = (x,0)$, and $s_1(x,t) = r(x,t) \in A \times I \cup X \times 0$, as required.

Theorem 10 *Suppose that (X,A) has the* HEP *and that $f_t\colon A \to B$ is a homotopy. Then the adjunction spaces $Y_0 = B \cup_{f_0} X$ and $Y_1 = B \cup_{f_1} X$ are homotopy equivalent rel B.*

Proof We show that Y_0 and Y_1 are both deformation retracts of $Y = B \cup_F (X \times I)$, where $F\colon A \times I \to B$ is the given homotopy. Consider the commutative diagram

$$\begin{tikzcd}
& X \times I \arrow[r] & Y \\
X \times 0 \arrow[r, "\subset"] & A \times I \cup X \times 0 \arrow[r] \arrow[u, "\subset"'] & Y_0 \arrow[u, "\subset"'] \\
A \times 0 \arrow[r, "\subset"] \arrow[u, "\subset"'] & A \times I \arrow[r, "F"] \arrow[u, "\subset"'] & B \arrow[u, "\subset"']
\end{tikzcd}$$

The two large rectangles are pushout squares by construction, and so is the lower left square. By the stacking lemma, all the squares are pushout squares. Since $A \times I \cup X \times 0$ is a deformation retract of $X \times I$ by Proposition 8, Y_0 is a deformation retract of Y. Similarly, so is Y_1. □

Cell complexes We need one basic example of the HEP.

Lemma 11 *The pair (D^n, S^{n-1}) has the* HEP.

Our proof may look different from the one in the book, but in fact is almost identical.

Proof Suppose given a homotopy $f_t\colon S^{n-1} \to Y$ and a map $g_0\colon D^n \to Y$ such that $g_0|S^{n-1} = f_0$. We assemble these to form a map $h\colon D_2 \to Y$ from the double-size disk $D_2 = \{x \in \mathbb{R}^n : \|x\| \le 2\}$ by setting

$$h(x) = \begin{cases} g_0(x) & \text{for } \|x\| \le 1; \\ f_{\|x\|-1}(x/\|x\|) & \text{for } 1 \le \|x\| \le 2. \end{cases}$$

The desired homotopy $g_t\colon D^n \to Y$ is then just $g_t(x) = h((1+t)x)$. □

Theorem 12 *Any* CW*-pair (X,A) has the* HEP.

Proof Given a homotopy $f_t\colon A \to Y$ and a map $g_0\colon X \to Y$ such that $g_0|A = f_0$, we construct a homotopy $g_t\colon X \to Y$ by induction on skeletons, $g_{t,n}\colon X^n \to Y$, starting from $X^{-1} = \emptyset$. Suppose we have $g_{t,n-1}$ that satisfies $g_{0,n-1} = g_0|X^{n-1}$ and $g_{t,n-1}|A^{n-1} = f_t|A^{n-1}$.

By definition, X^n may be obtained from X^{n-1} by attaching n-cells, one for each $\alpha \in \Lambda$, that have characteristic maps $\Phi_\alpha\colon (D^n_\alpha, S^{n-1}_\alpha) \to (X^n, X^{n-1})$ and attaching maps $\phi_\alpha = \Phi_\alpha|S^{n-1}_\alpha$, where each $(D^n_\alpha, S^{n-1}_\alpha)$ is a copy of (D^n, S^{n-1}). In other words, we have the pushout square

$$\begin{tikzcd}
\coprod_\alpha D^n_\alpha \arrow[r, "\Phi"] & X^n \\
\coprod_\alpha S^{n-1}_\alpha \arrow[r, "\phi"] \arrow[u, "\subset"'] & X^{n-1} \arrow[u, "\subset"']
\end{tikzcd}$$

where $\Phi|D^n_\alpha = \Phi_\alpha$ and $\phi|S^{n-1}_\alpha = \phi_\alpha$.

We use the pushout property to construct $g_{t,n}$ from $g_{t,n-1}$ and, for each α, a map $g_{t,\alpha}\colon D^n_\alpha \to Y$. Commutativity requires $g_{t,\alpha}|S^{n-1}_\alpha = g_{t,n-1} \circ \phi_\alpha$. To make $g_{t,n}$ satisfy $g_{0,n} = g_0|X^n$ and $g_{t,n}|A^n = f_t|A^n$, we need $g_{0,\alpha} = g_0 \circ \Phi_\alpha$, and, in case the α-th n-cell lies in A, $g_{t,\alpha} = f_t \circ \Phi_\alpha$.

So in the latter case, we are forced to take $g_{t,\alpha} = f_t \circ \Phi_\alpha$, which makes the other two conditions automatic, $f_t \circ \Phi_\alpha|S^{n-1}_\alpha = f_t \circ \phi_\alpha = g_{t,n-1} \circ \phi_\alpha$ and $g_{0,\alpha} = f_0 \circ \Phi_\alpha = g_0 \circ \Phi_\alpha$. Otherwise, we have only the maps $g_{0,\alpha} = g_0 \circ \Phi_\alpha\colon D^n_\alpha \to Y$ and $g_{t,n-1} \circ \phi_\alpha\colon S^{n-1}_\alpha \to Y$. These are consistent, as $g_{0,n-1} \circ \phi_\alpha = g_0 \circ \phi_\alpha = g_0 \circ \Phi_\alpha|S^{n-1}_\alpha = g_{0,\alpha}|S^{n-1}_\alpha$. As the pair $(D^n_\alpha, S^{n-1}_\alpha)$ has the HEP by Lemma 11, we can extend these two maps to $g_{t,\alpha}$ as required. This completes the induction step.

Finally, we define $g_t\colon X \to Y$ by $g_t|X^n = g_{t,n}$ for all n. $\square$