# Linear System Example: Repeated Real Root

We solve Example 2 of §56 in [Simmons, Second edition], on p. 431–2,

$$\begin{cases} \dfrac{dx}{dt} = 3x - 4y \\ \dfrac{dy}{dt} = x - y \end{cases}$$

by an alternate method. We apply the Laplace transform with the generic initial conditions $x(0) = k_1$ and $y(0) = k_2$ to get

$$\begin{cases} pX - k_1 = 3X - 4Y \\ pY - k_2 = X - Y \end{cases} \quad \text{or} \quad \begin{cases} (3-p)X - 4Y = -k_1 \\ X - (1+p)Y = -k_2. \end{cases}$$

We solve these simultaneous linear equations for $X$ and $Y$ by elimination,

$$\begin{cases} [4 \cdot 1 - (1+p)(3-p)]X = -4k_2 + (1+p)k_1 = (p+1)k_1 - 4k_2 \\ [-(3-p)(1+p) + 1 \cdot 4]Y = -(3-p)k_2 + k_1 = k_1 + (p-3)k_2. \end{cases}$$

The expressions in [ ] are both

$$4 - (1+p)(3-p) = 4 - (3 + 2p - p^2) = p^2 - 2p + 1 = (p-1)^2.$$

We therefore divide out by this and express everything in terms of $p - 1$, with an eye towards using the shift formula for Laplace transforms,

$$\begin{cases} X = \dfrac{(p+1)k_1 - 4k_2}{(p-1)^2} = \dfrac{(p-1)k_1 + 2k_1 - 4k_2}{(p-1)^2} = \dfrac{k_1}{p-1} + \dfrac{2k_1 - 4k_2}{(p-1)^2} \\ Y = \dfrac{k_1 + (p-3)k_2}{(p-1)^2} = \dfrac{(p-1)k_2 + k_1 - 2k_2}{(p-1)^2} = \dfrac{k_2}{p-1} + \dfrac{k_1 - 2k_2}{(p-1)^2} \end{cases}$$

Finally, we apply the inverse Laplace transform and find

$$\begin{cases} x = k_1 e^t + (2k_1 - 4k_2)te^t = [k_1 + (2k_1 - 4k_2)t]e^t \\ y = k_2 e^t + (k_1 - 2k_2)te^t = [k_2 + (k_1 - 2k_2)t]e^t. \end{cases}$$

This yields the solution (23) in the book if we take $k_1 = 2$ and $k_2 = 1$, and the solution (25) if we take $k_1 = 1$ and $k_2 = 0$. The correct values of $k_1$ and $k_2$ to use are obvious from the initial conditions. Moreover, we recover (26) if we put $c_1 = k_2$ and $c_2 = k_1 - 2k_2$.

Compare this treatment with the traditional one in the book to decide which is shorter or easier or more efficient. Draw your own conclusions.