Pushouts and Adjunction Spaces
This note augments material in Hatcher, Chapter 0.

**Pushouts** Given maps \( i : A \to X \) and \( f : A \to B \), we wish to complete the commutative square (a) in a canonical way.

![Diagram](image)

\[
\begin{align*}
\text{(a)} & \\
X & \xrightarrow{g} & Y \\
\uparrow i & & \uparrow j \\
A & \xrightarrow{f} & B
\end{align*}
\]

\[
\begin{align*}
\text{(b)} & \\
X & \xrightarrow{g} & Y & \xrightarrow{k} & Z \\
\uparrow i & & \uparrow j & & \uparrow h \\
A & \xrightarrow{f} & B & &
\end{align*}
\]

**Definition 2** We call (1)(a) a **pushout square** if it commutes, \( j \circ f = g \circ i \), and is **universal**, in the sense that given any space \( Z \) and maps \( h : X \to Z \) and \( k : B \to Z \) such that \( k \circ f = h \circ i \), there exists a unique map \( m : Y \to Z \) that makes diagram (1)(b) commute, \( m \circ g = h \) and \( m \circ j = k \). We then call \( Y \) a **pushout** of \( i \) and \( f \).

The good news is that uniqueness of pushouts is automatic.

**Proposition 3** Given any maps \( i \) and \( f \) as in (1)(a), the pushout space \( Y \) is unique up to canonical homeomorphism.

**Proof** Suppose \( Y' \), with maps \( g' : X \to Y' \) and \( j' : B \to Y' \), is another pushout. Take \( Z = Y' \); we find a map \( m : Y \to Y' \) such that \( m \circ g = g' \) and \( m \circ j = j' \). By reversing the roles of \( Y \) and \( Y' \), we find \( m' : Y' \to Y \) such that \( m' \circ g' = g \) and \( m' \circ j' = j \). Then \( m' \circ m \circ g = m' \circ g' = g \), and similarly \( m' \circ m \circ j = j \). Now take \( Z = Y \), \( h = g \), and \( k = j \). We have two maps, \( m' \circ m : Y \to Y \) and \( \text{id}_Y : Y \to Y \), that make diagram (1)(b) commute; by the uniqueness in Definition 2, \( m' \circ m = \text{id}_Y \). Similarly, \( m \circ m' = \text{id}_Y \), so that \( m \) and \( m' \) are inverse homeomorphisms. \( \Box \)

However, existence is not automatic; pushouts must be constructed.

**Proposition 4** Let \( i : A \to X \) and \( f : A \to B \) be any maps. Then there exists a pushout \( Y \) as in Definition 2.

**Proof** Let \( \sim \) be the smallest equivalence relation on the topological disjoint union \( X \coprod B \) that satisfies \( i(a) \sim f(a) \) for all \( a \in A \). [It is the intersection of all equivalence relations on \( X \coprod B \) that have this property.] We take \( Y \) as the quotient space \( (X \coprod B)/\sim \), with quotient map \( q : X \coprod B \to Y \), and set \( g = q|X \) and \( j = q|B \).

We have commutativity, since for any \( a \in A \), \( g(i(a)) = q(i(a)) = q(f(a)) = j(f(a)) \). Given \( h \) and \( k \) as in diagram (1)(b), we define \( \tilde{m} : X \coprod B \to Z \) by \( \tilde{m}|X = h \) and \( \tilde{m}|B = k \). Then for any \( a \in A \), \( \tilde{m}(i(a)) = h(i(a)) = k(f(a)) = \tilde{m}(f(a)) \). It follows that \( \tilde{m} \) is constant on each equivalence class and hence factors through the map \( g : X \coprod B \to Y \) to yield the desired map \( m : Y \to Z \). Further, \( m \) is unique because it is required to satisfy \( m \circ g = \tilde{m} \), with \( \tilde{m} \) defined as above. \( \Box \)

**Corollary 5** A subset \( V \subset Y \) is open (resp. closed) if and only if \( g^{-1}(V) \) is open (resp. closed) in \( X \) and \( j^{-1}(V) \) is open (resp. closed) in \( B \). \( \Box \)
We can stack pushout squares. The proof depends only on the universal property in Definition 2 and is omitted. (Try it!)

**Proposition 6** Suppose given a commutative diagram

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\uparrow & & \uparrow \\
A & \rightarrow & B
\end{array}
\quad
\begin{array}{ccc}
 & Z \\
\uparrow & & \uparrow \\
 & C
\end{array}
\]

in which \(ABXY\) is a pushout square. Then \(ACXZ\) is a pushout square if and only if \(BCYZ\) is a pushout square. \(\square\)

**Remark** The third possible implication fails: if \(ACXZ\) and \(BCYZ\) are pushout squares, \(ABXY\) need not be one. For a simple example, take \(A = X = Y = C = Z\) to be a point, and \(B\) any space with more than one point.

We can also take products.

**Proposition 7** If diagram \((1)(a)\) is a pushout square and \(W\) is a locally compact space, then

\[
\begin{array}{ccc}
X \times W & \rightarrow & Y \times W \\
\uparrow & & \uparrow \\
A \times W & \rightarrow & B \times W
\end{array}
\]

is another pushout square.

**Proof** This follows from the standard (but non-trivial) topological result that if \(q: X \coprod B \rightarrow Y\) is a quotient map, so is \(q \times \text{id}_W: (X \coprod B) \times W \rightarrow Y \times W\). [This statement is false without some condition on \(W\).] \(\square\)

**Corollary 8** “The pushout of homotopies is a homotopy.” Given the pushout square \((1)(a)\) and homotopies \(h_t: X \rightarrow Z\) and \(k_t: B \rightarrow Z\) such that \(h_t \circ i = k_t \circ f\) for all \(t\), define \(m_t: Y \rightarrow Z\) by \(m_t \circ g = h_t\) and \(m_t \circ j = k_t\); then \(m_t\) is a homotopy.

**Proof** We take \(W = I\) in Proposition 7. \(\square\)

**Adjunction spaces** The bad news about pushouts is that \(Y\), being a quotient space, is in general poorly behaved. Even if \(A\), \(B\) and \(X\) are very nice spaces, \(Y\) need not even be Hausdorff. In the construction, it is far from clear what the equivalence classes in \(X \coprod B\) are, or what the points of \(Y\) really are.

We shall say no more at this level of generality. From now on, we limit attention to the following special case.

**Proposition 9** In the pushout square \((1)(a)\), if \(i\) is a closed embedding, so is \(j\).

**Proof** Under this hypothesis, it becomes clear what the equivalence classes in \(X \coprod B\) are: they are the singletons \(\{x\}\) for each \(x \in X - A\), and the sets \(i(f^{-1}(b)) \coprod \{b\}\) for
each \( b \in B \). Thus as a set, \( Y \) is the disjoint union of \( X - A \) and \( B \); in particular, \( j \) is injective. However, the topology on \( Y \) is not the disjoint union topology.

Recall that to prove \( j \) is a closed embedding, it is only necessary to show that \( j(F) \) is closed in \( Y \) whenever \( F \) is closed in \( B \). Because \( j \) is injective, \( j^{-1}(j(F)) = F \) and \( g^{-1}(j(F)) = i(f^{-1}(F)) \). By Corollary 5, \( j(F) \) is closed. \[ \square \]

Henceforth, we simplify notation by assuming that \( A \) and \( B \) actually are closed subspaces of \( X \) and \( Y \), and we (usually) suppress \( i \) and \( j \). Commutativity is simply expressed by \( g|A = f \), and we have a map of pairs \( g: (X, A) \to (Y, B) \). Informally, we obtain \( Y \) from \( B \) by gluing \( X \) to \( B \) along the subspace \( A \) of \( X \) as directed by the map \( f \); we identify each point \( a \in A \) with its image \( f(a) \in B \).

**Definition 10** When \( A \subset X \) is a closed subspace, we call \( Y \) an adjunction space and \( f: A \to B \) the attaching map. We write \( Y = B \cup_f X \) (or \( B \sqcup_f X \) in Hatcher).

**Example** If the subspaces \( A \) and \( B \) of a space \( X \) are both open or both closed, then

\[
\begin{array}{ccc}
A & \xrightarrow{c} & A \cup B \\
\uparrow & & \uparrow \\
A \cap B & \xrightarrow{c} & B \\
\end{array}
\]

is a pushout square. This is simply a restatement of the standard result that given a function \( f: A \cup B \to Y \), if \( f|A \) and \( f|B \) are continuous, then \( f \) itself is continuous.

In general (though not always), \( j \) inherits properties from \( i \) and \( g \) from \( f \).

**Proposition 11** Assume diagram (1)(a) is an adjunction square with \( A \subset X \) a closed subspace. Then:

(a) If \( A = X \), then \( B = Y \);
(b) \( g|(X - A): X - A \to Y - B \) is a homeomorphism;
(c) If \( B \) and \( X \) are \( T_1 \) spaces, so is \( Y \);
(d) If \( B \) and \( X \) are normal spaces, so is \( Y \);
(e) If \( F \) is a closed subspace of \( X \) with \( F \cap A = \emptyset \), then \( g|F: F \to Y \) is a closed embedding;
(f) If \( A \) is a retract of \( X \), then \( B \) is a retract of \( Y \);
(g) If \( (X, A) \) satisfies the homotopy extension property, so does \( (Y, B) \);
(h) If \( A \) is a deformation retract of \( X \), then \( B \) is a deformation retract of \( Y \).

**Proof** By now, (a) is trivial.

In (b), the map is obviously a continuous bijection. To see that it is an open map, take an open set \( V \) in \( X - A \); then \( g(V) \cap B = \emptyset \) and \( g^{-1}(g(V)) = V \) show that \( g(V) \) is open. Similarly for (e).

For (c), the equivalence classes in \( X \| B \) in Proposition 9 are obviously closed.

For (d), it is convenient to understand “normal” as not implying \( T_1 \); then the Tietze Extension Theorem can be restated as: \( Y \) is normal if and only if any map \( u: G \to \mathbb{R} \) from a closed subspace \( G \) of \( Y \) extends to a map \( v: Y \to \mathbb{R} \).
Given \( u \), because \( B \) is normal, the map \( u|(B \cap G) \) extends to a map \( v_B: B \to \mathbb{R} \). Now we put \( F = g^{-1}(G) \) and work in \( X \). The two maps \( v_B \circ f: A \to \mathbb{R} \) and \( u \circ (g|F): F \to \mathbb{R} \) agree on \( A \cap F \) and so define a map \( A \cup F \to \mathbb{R} \). Because \( X \) is normal, this extends to a map \( v_X: X \to \mathbb{R} \). Since \( v_X|A = v_B \circ f \), we find a map \( v: Y \to \mathbb{R} \) that satisfies \( v \circ g = v_X \) and \( v|B = v_B \). By construction, \( v \) extends \( u \).

In (f), suppose \( r: X \to A \) is a retraction, so that \( r|A = \text{id}_A \). We define the retraction \( s: Y \to B \) by \( s \circ g = f \circ r \) and \( s|B = \text{id}_B \).

In (g), suppose given a homotopy \( k_t: B \to Z \) and a map \( m_0: Y \to Z \) such that \( m_0|B = k_0 \). We have a homotopy \( k_t \circ f: A \to Z \) and a map \( m_0 \circ g: X \to Z \) such that \( (m_0 \circ g)|A = m_0 \circ g \circ i = m_0 \circ j \circ f = k_0 \circ f; \) by the HEP for \((X, A)\), there is a homotopy \( h_t: X \to Z \) such that \( h_0 = m_0 \circ g \) and \( h_t|A = k_t \circ f \). For each \( t \), define \( m_t: Y \to Z \) by \( m_t \circ g = h_t \) and \( m_t|B = k_t \). By Corollary 8, this is the desired homotopy.

In (h), let \( d_t: X \to X \) be a deformation retraction, so that \( d_t|A = i \), \( d_0 = \text{id}_X \), and \( d_1 = r \). We use the homotopy \( h_t = g \circ d_t: X \to Y \) and the constant homotopy \( k_t = j: B \to Y \) to construct \( m_t: Y \to Y \), which is a homotopy by Corollary 8. We see that by uniqueness, \( m_0 = \text{id}_Y \) and \( m_1 = s \), the retraction in (f).

**Examples of adjunction spaces**

1. The quotient space \( X/A \) is obtained by taking \( B \) to be a one-point space. As a set, its points are those of \( X - A \) together with one point corresponding to \( A \).

2. The wedge \( X \vee Y \) of two spaces \( X \) and \( Y \) with basepoints \( x_0 \) and \( y_0 \) is the quotient space \( (X \sqcup Y)/\{x_0, y_0\} \) obtained from the disjoint union \( X \sqcup Y \) by identifying the two basepoints. (It is often defined as the subspace \( X \times y_0 \cup x_0 \times Y \) of \( X \times Y \); it is easy to construct homeomorphisms between these two definitions.)

More generally, one can form the wedge \( \bigvee_{\alpha} X_\alpha \) of any collection of based spaces \( (X_\alpha, x_\alpha) \) as the quotient space \( (\coprod_{\alpha} X_\alpha)/\prod_{\alpha} x_\alpha \).

3. The cone \( CX \) on \( X \) is the quotient \((X \times I)/(X \times 0)\). By (e), it contains a copy of \( X \) as the image of \( X \times 1 \).

4. The suspension \( SX \) of \( X \) is the pushout of \( X \times \partial I \subset X \times I \) and the projection \( X \times \partial I \to \partial I = \{0, 1\} \). It contains a copy of \( X \) as the image of \( X \times (1/2) \).

5. The mapping cylinder \( M_f \) of \( f: A \to B \) is obtained by taking \( X = A \times I \), with inclusion \( A \cong A \times 1 \subset A \times I \). By (h), \( B \) is a deformation retract of \( M_f \). Also, by (e), \( M_f \) contains a copy of \( A \) as the image of \( A \times 0 \), as well as \( B \).

6. The mapping cone \( C_f \) of \( f: A \to B \) is obtained by taking \( X = CA \), with the inclusion \( A \subset CA \), or equivalently as \( M_f/A \) (with the help of Proposition 6).

7. If we take \( X = D^n \), the closed unit \( n \)-disk in \( \mathbb{R}^n \), and \( A = S^{n-1} \), its boundary sphere, the resulting space \( Y = B \cup_f D^n \), commonly written \( Y = B \cup_e e^n \), is said to be obtained from \( B \) by attaching an \( n \)-cell, using the attaching map \( f: S^{n-1} \to B \). Then \( g: (D^n, S^{n-1}) \to (Y, B) \) is called the characteristic map of the \( n \)-cell.

One can attach many \( n \)-cells by taking \( X = \coprod_{\alpha} D_{\alpha}^n \) and \( A = \coprod_{\alpha} S_{\alpha}^{n-1} \), using attaching maps \( f_{\alpha}: S_{\alpha}^{n-1} \to B \), where each \( D_{\alpha}^n \) is a copy of \( D^n \), with boundary \( S_{\alpha}^{n-1} \).

8. The smash product \( X \wedge Y \) is the quotient space \((X \times Y)/(X \vee Y)\).

9. The join \( X \ast Y \) is the pushout of \( X \times \partial I \times Y \subset X \times I \times Y \) and the map \( X \times \partial I \times Y \to X \vee Y \) formed from the projections \( X \times 0 \times Y \to X \) and \( X \times 1 \times Y \to Y \).