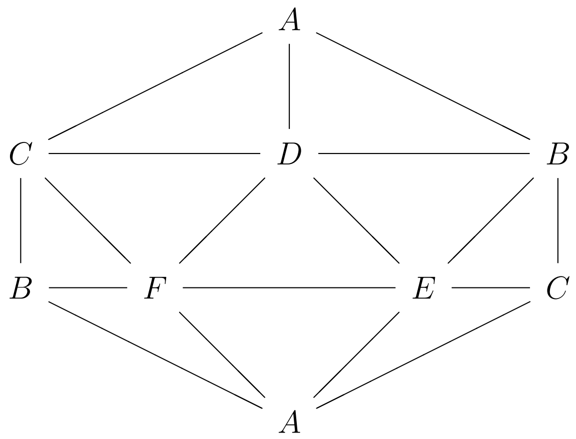
The Real Projective Plane Triangulated

Triangulation The most efficient way to triangulate the real projective plane $\mathbb{R}P^2$ is the following:

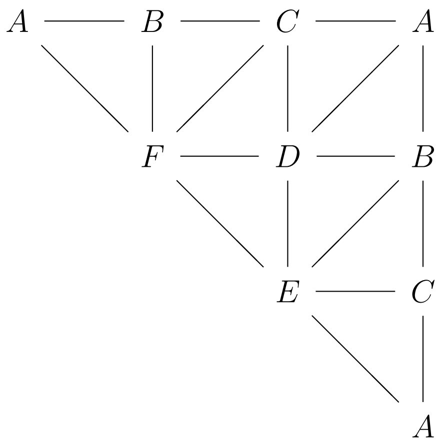


(1)

This is a hexagon with diametrically opposite points identified. All triangles are filled in.

Connected sum Simplicially, the *connected sum* $S\#T$ of two triangulated surfaces S and T is obtained by deleting one 2-simplex from each and identifying the corresponding faces in S and T of these two simplices. (There are choices involved, but they do not affect the result, in dimension two.)

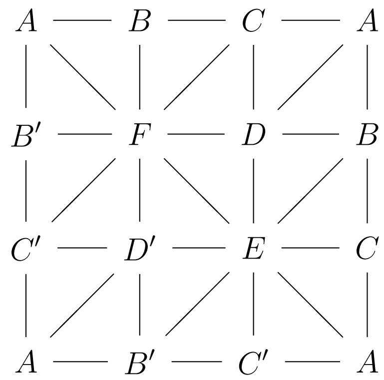
For example, to construct $\mathbb{R}P^2\#\mathbb{R}P^2$, we may start by removing the 2-simplex AEF from (1) and redrawing to obtain the triangle



(2)

in which two of the edges are identified. The diagonal edge, $AFEA$, is the remaining boundary circle. You can check that all the necessary simplices are there and that no extra ones crept in.

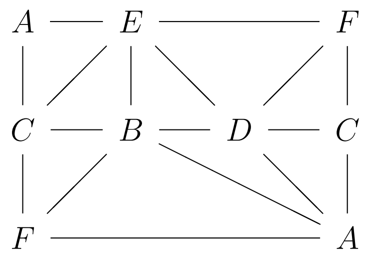
To form the connected sum, we take a second copy of $\mathbb{R}P^2$, with vertices A', B' etc., except that we identify $A' = A$, $E' = E$, and $F' = F$. We simply reflect (2) in the diagonal and relabel to obtain the square



(3)

in which the top edge is identified with the right edge and the left edge is identified with the bottom edge.

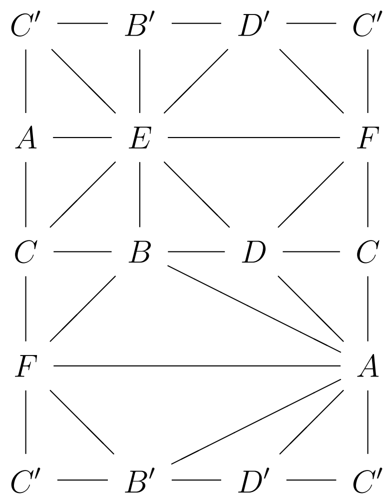
The Möbius band To recognize $\mathbb{R}P^2$ with a 2-simplex removed as the Möbius band, we redraw (2) as the following rectangle



(4)

in which the left and right edges are to be identified, with a twist.

The Klein bottle To see that $\mathbb{R}P^2 \# \mathbb{R}P^2$ is indeed the Klein bottle, we start from the Möbius band (4) and attach another one, corresponding to the second copy of $\mathbb{R}P^2$. However, we do this in a non-obvious way, by distributing the second Möbius band above and below (4) to obtain the rectangle



(5)

Here, the top and bottom edges are identified, and the left and right edges are identified with a twist.