Universal Coefficient Theorem for Homology

We present a direct proof of the universal coefficient theorem for homology that is simpler and shorter than the standard proof.

THEOREM 1 *Given a chain complex C in which each C_n is free abelian, and a coefficient group G, we have for each n the natural short exact sequence*

$$0 \longrightarrow H_n(C) \otimes G \longrightarrow H_n(C \otimes G) \longrightarrow \mathrm{Tor}(H_{n-1}(C), G) \longrightarrow 0, \qquad (2)$$

which splits (but not naturally).

In particular, this applies immediately to singular homology.

THEOREM 3 *Given a pair of spaces (X, A) and a coefficient group G, we have for each n the natural short exact sequence*

$$0 \longrightarrow H_n(X, A) \otimes G \longrightarrow H_n(X, A; G) \longrightarrow \mathrm{Tor}(H_{n-1}(X, A), G) \longrightarrow 0,$$

which splits (but not naturally). □

We shall derive diagram (2) as an instance of the following elementary result.

LEMMA 4 *Given homomorphisms $f\colon K \to L$ and $g\colon L \to M$ of abelian groups, with a splitting homomorphism $s\colon L \to K$ such that $f \circ s = \mathrm{id}_L$, we have the split short exact sequence*

$$0 \longrightarrow \mathrm{Ker}\, f \xrightarrow{\subset} \mathrm{Ker}(g \circ f) \xrightarrow{f'} \mathrm{Ker}\, g \longrightarrow 0, \qquad (5)$$

where $f' = f|\,\mathrm{Ker}(g \circ f)$, with the splitting $s' = s|\,\mathrm{Ker}\, g\colon \mathrm{Ker}\, g \to \mathrm{Ker}(g \circ f)$.

Proof We note that f' and s' are defined, as $f(\mathrm{Ker}(g \circ f)) \subset \mathrm{Ker}\, g$ and $s(\mathrm{Ker}\, g) \subset \mathrm{Ker}(g \circ f)$. (In detail, if $l \in \mathrm{Ker}\, g$, $(g \circ f)sl = gfsl = gl = 0$ shows that $sl \in \mathrm{Ker}(g \circ f)$.) Then $f \circ s = \mathrm{id}_L$ restricts to $f' \circ s' = \mathrm{id}$. Since $\mathrm{Ker}\, f \subset \mathrm{Ker}(g \circ f)$, we have $\mathrm{Ker}\, f' = \mathrm{Ker}\, f \cap \mathrm{Ker}(g \circ f) = \mathrm{Ker}\, f$. This completes the proof. □

Preliminaries We consider a chain complex C as in Theorem 1. We adopt the usual notation: Z_n for the group of n-cycles, B_n for the group of n-boundaries, and $H_n = H_n(C) = Z_n/B_n$; also $Z_n(C \otimes G)$ etc. for the chain complex $C \otimes G$. The key idea of our proof of Theorem 1 is to express $\partial\colon C_n \to C_{n-1}$ as the composite

$$\partial\colon C_n \longrightarrow \frac{C_n}{B_n} \longrightarrow \frac{C_n}{Z_n} \xrightarrow{\cong} B_{n-1} \xrightarrow{\subset} Z_{n-1} \xrightarrow{\subset} C_{n-1}. \qquad (6)$$

By definition, we have the short exact sequence

$$0 \longrightarrow B_n \xrightarrow{\subset} Z_n \longrightarrow H_n \longrightarrow 0 \qquad (7)$$

for any n. As B_n and Z_n are subgroups of the free abelian group C_n and therefore free abelian, we recognize (7) as a free resolution of H_n.

Also by definition, from diagram (6) we have the short exact sequence

$$0 \longrightarrow Z_n \xrightarrow{\subset} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0, \qquad (8)$$

which splits because B_{n-1} is free abelian. Using (7) and (8), we may rewrite the canonical short exact sequence

$$0 \longrightarrow Z_n/B_n \xrightarrow{\subset} C_n/B_n \longrightarrow C_n/Z_n \longrightarrow 0$$

as

$$0 \longrightarrow H_n \longrightarrow C_n/B_n \xrightarrow{\bar\partial} B_{n-1} \longrightarrow 0, \tag{9}$$

where $\bar\partial$ is a quotient of ∂. As B_{n-1} is free abelian, this short exact sequence also splits, and we may choose a splitting homomorphism $s\colon B_{n-1} \to C_n/B_n$.

Proof of Theorem 1 We tensor diagram (6) and the above short exact sequences with G, to form the following diagram of exact sequences.

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H_n \otimes G & & & & \\
 & & & & \downarrow & & & & \\
 & & B_n \otimes G & \longrightarrow & C_n \otimes G \longrightarrow \dfrac{C_n}{B_n} \otimes G & \longrightarrow & 0 & & 0 \\
 & & & & s\otimes\mathrm{id} \uparrow \;\; \downarrow \bar\partial\otimes\mathrm{id} & & & & \downarrow \\
 0 & \longrightarrow & \mathrm{Tor}(H_{n-1}, G) & \longrightarrow & B_{n-1} \otimes G & \longrightarrow & & & Z_{n-1} \otimes G \\
 & & & & \downarrow & \searrow & & & \downarrow \\
 & & & & 0 & & & & C_{n-1} \otimes G
\end{array} \tag{10}$$

Both vertical sequences are exact, since (9) and (8) split. The upper horizontal sequence is exact, because $- \otimes G$ is right exact. The lower horizontal sequence is exact, as $\mathrm{Tor}(H_{n-1}, G) = \mathrm{Ker}[B_{n-1} \otimes G \to Z_{n-1} \otimes G]$ by definition.

We deduce that $B_{n-1}(C \otimes G) = \mathrm{Im}(\partial \otimes \mathrm{id}) = \mathrm{Im}[B_{n-1} \otimes G \to C_{n-1} \otimes G]$. On replacing $n-1$ by n, we see that

$$B_n(C \otimes G) = \mathrm{Im}[B_n \otimes G \longrightarrow C_n \otimes G] = \mathrm{Ker}[C_n \otimes G \longrightarrow (C_n/B_n) \otimes G].$$

Then $(C_n \otimes G)/B_n(C \otimes G) \cong (C_n/B_n) \otimes G$. Also, $Z_n(C \otimes G) = \mathrm{Ker}[C_n \otimes G \to Z_{n-1} \otimes G]$, and hence $(C_n \otimes G)/Z_n(C \otimes G) \cong \mathrm{Im}[C_n \otimes G \to Z_{n-1} \otimes G] \subset Z_{n-1} \otimes G$.

We next use the canonical short exact sequence

$$0 \longrightarrow \frac{Z_n(C \otimes G)}{B_n(C \otimes G)} \xrightarrow{\subset} \frac{C_n \otimes G}{B_n(C \otimes G)} \longrightarrow \frac{C_n \otimes G}{Z_n(C \otimes G)} \longrightarrow 0$$

to rewrite our target group as

$$H_n(C \otimes G) = \frac{Z_n(C \otimes G)}{B_n(C \otimes G)} \cong \mathrm{Ker}\left[\frac{C_n}{B_n} \otimes G \longrightarrow Z_{n-1} \otimes G\right].$$

We now apply Lemma 4 with $K = (C_n/B_n) \otimes G$, $L = B_{n-1} \otimes G$ and $M = Z_{n-1} \otimes G$, and use the splitting $s \otimes \mathrm{id}\colon L \to K$. Diagram (10) identifies $\mathrm{Ker}[K \to L]$ with $H_n \otimes G$ and $\mathrm{Ker}[L \to M]$ with $\mathrm{Tor}(H_{n-1}, G)$. These identifications reduce the split short exact sequence (5) to the desired (2). □