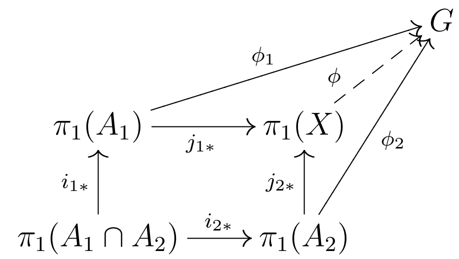
van Kampen's Theorem

We present a variant of Hatcher's proof of van Kampen's Theorem, for the simpler case of just two open sets.

THEOREM 1 *Let X be a space with basepoint x_0. Let A_1 and A_2 be open subspaces that contain x_0 and satisfy $X = A_1 \cup A_2$. Assume that A_1, A_2 and $A_1 \cap A_2$ (and hence X) are all path-connected. Then the commutative square of homomorphisms induced by inclusions in the diagram*



(2)

is a pushout square of groups: given any group G and homomorphisms $\phi_1: \pi_1(A_1) \to G$ and $\phi_2: \pi_1(A_2) \to G$ such that $\phi_1 \circ i_{1*} = \phi_2 \circ i_{2*}$, there is a unique homomorphism $\phi: \pi_1(X) \to G$ that makes the diagram commute, i. e. $\phi \circ j_{1*} = \phi_1$ and $\phi \circ j_{2*} = \phi_2$.

The proof will involve subdivisions of paths. Take any path $f: I \to X$ in X and let $0 = s_0 < s_1 < s_2 < \ldots < s_{m-1} < s_m = 1$ be any subdivision of I. Write f_i for the obvious path from $f(s_{i-1})$ to $f(s_i)$, namely $f_i(s) = f((1-s)s_{i-1} + ss_i)$.

LEMMA 3 *With the above notation, there is a path homotopy $f \simeq f_1 \cdot f_2 \cdot \ldots \cdot f_m$.*

Remark The convention here is that parentheses are to be inserted anywhere in $f_1 \cdot f_2 \cdot \ldots \cdot f_m$ to make it defined; since all choices yield path-homotopic results, the specific choice is irrelevant.

Proof Given $a, b \in I$, denote by $\lambda_{a,b}: I \to I$ the straightline path in I from a to b, i. e. $\lambda_{a,b}(s) = (1-s)a + sb$. Then $f_i = f \circ \lambda_{s_{i-1},s_i}$ and

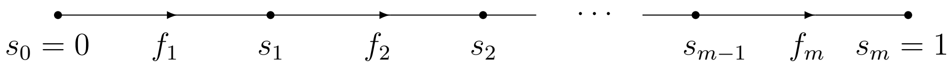
$$\begin{aligned} f_1 \cdot f_2 \cdot \ldots \cdot f_m &= (f \circ \lambda_{s_0,s_1}) \cdot (f \circ \lambda_{s_1,s_2}) \cdot \ldots \cdot (f \circ \lambda_{s_{m-1},s_m}) \\ &= f \circ (\lambda_{s_0,s_1} \cdot \lambda_{s_1,s_2} \cdot \ldots \cdot \lambda_{s_{m-1},s_m}) \\ &\simeq f \circ \mathrm{id}_I = f, \end{aligned}$$

since $\lambda_{s_0,s_1} \cdot \lambda_{s_1,s_2} \cdot \ldots \cdot \lambda_{s_{m-1},s_m} \simeq \mathrm{id}_I$, I being convex. □

Proof of Theorem Because we have to deal with four different spaces, we write $[f]_Y$ for the path-homotopy class in Y of a loop in Y. (The basepoint for all loops will be x_0 and is suppressed from the notation.)

We have to construct $\phi(f)$ for each loop f in X at x_0.

The recipe Given f, the open sets $f^{-1}(A_1)$ and $f^{-1}(A_2)$ cover the compact metric space I. By the Lebesgue covering lemma, there is a subdivision $0 = s_0 < s_1 < s_2 < \ldots < s_m = 1$ of I such that for each i, $f([s_{i-1}, s_i]) \subset A_{\alpha(i)}$, where $\alpha(i) = 1$ or 2.



We note that $f(s_i) \in A_{\alpha(i)} \cap A_{\alpha(i+1)}$. Since this space is path-connected, we can choose, for each i $(0 < i < n)$, a path g_i in $A_{\alpha(i)} \cap A_{\alpha(i+1)}$ from x_0 to $f(s_i)$; we also put $g_0 = g_m = c$, the constant loop at x_0. For each i, we have the loop $\omega_i = g_{i-1} \cdot f_i \cdot \bar{g}_i$ in $A_{\alpha(i)}$. We put $z_i = \phi_{\alpha(i)}([\omega_i]_{A_{\alpha(i)}})$ and set

$$\phi(f) = z_1 z_2 \dots z_m \qquad \text{in } G. \tag{4}$$

Commutativity Given a loop f in A_β, we take $m = 1$ and don't subdivide at all, $0 = s_0 < s_1 = 1$. Then $\omega_1 \simeq f$ and (4) reduces to $\phi(f) = \phi_\beta([f]_{A_\beta})$, which will imply that $\phi(j_{\beta*}([f]_{A_\beta})) = \phi(f) = \phi_\beta([f]_{A_\beta})$, as required.

Uniqueness of $\phi[f]$ By Lemma 3,

$$[\omega_1]_X[\omega_2]_X \dots [\omega_m]_X = [c \cdot f_1 \cdot \bar{g}_1 \cdot g_1 \cdot f_2 \cdot \bar{g}_2 \cdot \dots \cdot g_{m-1} \cdot f_m \cdot c]_X = [f]_X$$

in $\pi_1(X)$. Since ϕ is to be a homomorphism, we must use the formula (4), with

$$z_i = \phi([\omega_i]_X) = \phi(j_{\alpha(i)*}([\omega_i]_{A_{\alpha(i)}})) = \phi_{\alpha(i)}([\omega_i]_{A_{\alpha(i)}}).$$

Homomorphism To verify that ϕ is a homomorphism on $\pi_1(X)$, we have to show that, given two loops f and f', $\phi(f \cdot f') = \phi(f)\phi(f')$. Choose subdivisions, $\alpha(i)$ and g_i as above for each of f and f', to get elements z_i and z'_j of G. If we combine these subdivisions to get a subdivision for $f \cdot f'$, we find

$$\phi(f \cdot f') = z_1 z_2 \dots z_m z'_1 z'_2 \dots z'_{m'} = \phi(f)\phi(f').$$

To prove that $\phi[f]$ is well defined on $\pi_1(X)$ (which will complete the proof), we have to show it is independent of:

(i) The choice of $\alpha(i)$ for each i;
(ii) The choice of g_i for each i;
(iii) The choice of subdivision of I;
(iv) The choice of f in the homotopy class $[f]$.

$\phi(f)$ is independent of the choice of $\alpha(i)$. There is no choice unless f_i lies in $A_1 \cap A_2$. In this case, ω_i also lies in $A_1 \cap A_2$ and z_i is well defined, because

$$\phi_1([\omega_i]_{A_1}) = \phi_1(i_{1*}([\omega_i]_{A_1 \cap A_2})) = \phi_2(i_{2*}([\omega_i]_{A_1 \cap A_2})) = \phi_2([\omega_i]_{A_2}).$$

$\phi(f)$ is independent of g_i. Suppose we choose a different path g'_i from x_0 to $f(s_i)$ in $A_\beta \cap A_\gamma$, where $\alpha(i) = \beta$ and $\alpha(i+1) = \gamma$. We then have the loop $k = g_i \cdot \bar{g}'_i$, so that $\bar{k} = g'_i \cdot \bar{g}_i$. We replace ω_i by $\omega'_i = g_{i-1} \cdot f_i \cdot \bar{g}'_i \simeq \omega_i \cdot k$ and ω_{i+1} by $\omega'_{i+1} = g'_i \cdot f_{i+1} \cdot \bar{g}_{i+1} \simeq \bar{k} \cdot \omega_{i+1}$. The other loops ω_j are unchanged. Then

$$z'_i z'_{i+1} = z_i \phi_\beta([k]_{A_\beta}) \phi_\gamma([\bar{k}]_{A_\gamma}) z_{i+1} \qquad \text{in } G.$$

Since $[\bar{k}] = [k]^{-1}$, this reduces to $z_i z_{i+1}$ if $\beta = \gamma$. Otherwise, we use

$$\phi_1([k]_{A_1}) = \phi_1(i_{1*}([k]_{A_1 \cap A_2})) = \phi_2(i_{2*}([k]_{A_1 \cap A_2})) = \phi_2([k]_{A_2}).$$

$\phi(f)$ is independent of the subdivision. To compare any two subdivisions, it is enough to consider the effect of inserting *one* extra point s_+ between s_{i-1} and s_i. Suppose

f_i maps into A_β; then we break up $f_i \simeq f_i' \cdot f_i''$ as in Lemma 3, where f_i' and f_i'' also map into A_β. We choose a path g_+ in A_β from x_0 to $f(s_+)$; then in (4), we replace z_i by

$$\begin{aligned} z_i' z_i'' &= \phi_\beta([g_{i-1} \cdot f_i' \cdot \bar{g}_+]_{A_\beta})\phi_\beta([g_+ \cdot f_i'' \cdot \bar{g}_i]_{A_\beta}) \\ &= \phi_\beta([g_{i-1} \cdot f_i' \cdot \bar{g}_+ \cdot g_+ \cdot f_i'' \cdot \bar{g}_i]_{A_\beta}) \\ &= \phi_\beta([\omega_i]_{A_\beta}) = z_i, \end{aligned}$$

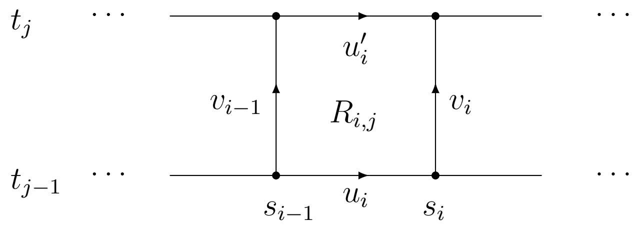
which does not change $\phi(f)$.

Homotopy To make ϕ well defined on $\pi_1(X)$, we have to show that for any loop homotopy f_t, $\phi(f_0) = \phi(f_1)$. The homotopy provides a map $F: I \times I \to X$. The open sets $F^{-1}(A_1)$ and $F^{-1}(A_2)$ cover the compact metric space $I \times I$. Again by the Lebesgue covering lemma, we can subdivide the path variable $s \in I$ by

$$0 = s_0 < s_1 < \ldots < s_m = 1 \tag{5}$$

and the homotopy variable $t \in I$ by $0 = t_0 < t_1 < \ldots < t_n = 1$ to make each rectangle $R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ map into A_1 or into A_2; we have only to make them small enough. It is enough to show that $\phi(f_{t_{j-1}}) = \phi(f_{t_j})$ for each j.

Suppose $F(R_{i,j}) \subset A_\beta$; we use $\alpha(i) = \beta$ and the same subdivision (5) for both $f_{t_{j-1}}$ and f_{t_j}, as pictured.



As in Lemma 3, we subdivide $f_{t_{j-1}} \simeq u_1 \cdot u_2 \cdot \ldots \cdot u_m$ and $f_{t_j} \simeq u_1' \cdot u_2' \cdot \ldots \cdot u_m'$. Let v_i be the obvious path from $F(s_i, t_{j-1})$ to $F(s_i, t_j)$. As usual, we choose paths g_i from x_0 to $f_{t_{j-1}}(s_i)$ for $f_{t_{j-1}}$; but for f_{t_j}, we use the paths $g_i' = g_i \cdot v_i$. Then for (4), $\omega_i = g_{i-1} \cdot u_i \cdot \bar{g}_i$, while $\omega_i' = g_{i-1} \cdot v_{i-1} \cdot u_i' \cdot \bar{v}_i \cdot \bar{g}_i$. Because $R_{i,j}$ is convex and maps into A_β, $u_i \simeq v_{i-1} \cdot u_i' \cdot \bar{v}_i$ in A_β, which implies $\omega_i' \simeq \omega_i$ and $z_i' = z_i$. □

Extensions 1. The sets A_1 and A_2 do not need to be open; it is enough to have $\operatorname{Int} A_1 \cup \operatorname{Int} A_2 = X$. To prove existence of a suitable subdivision of I or $I \times I$, we start from the open covering of X by $\operatorname{Int} A_1$ and $\operatorname{Int} A_2$. The rest of the proof is unchanged.

2. Instead of two open sets A_1 and A_2, we can have many, even infinitely many. Beyond the obvious changes to the statement and proof, one extra condition is needed:

$$A_\alpha \cap A_\beta \cap A_\gamma \text{ is path-connected for all } \alpha,\ \beta,\ \gamma.$$

Hatcher (p. 44) gives an example to show the necessity. See if you can find where in this proof it is needed.