

Final Examination

Three hours. Closed book. No notes. No calculators.

200 points, 20 per question.

Partial credit may be available, but only if you show your working.

Begin each of the ten questions on a new page and number it clearly in the margin.

If you use two books, label them "Book 1 of 2" and "Book 2 of 2". (If you use three books, ...)

Make sure your T.A.'s name is on each book, as well as your name.

Do not use decimal evaluations of square roots, trigonometric functions and such.

All integrals have been well cooked to work out easily.

Use only the officially provided blue books.

1. (a) State Green's Theorem.
- (b) Use Green's Theorem to evaluate the contour integral

$$\int_C (1 + y^8)dx + (x^2 + e^y)dy$$

where C denotes the boundary of the region enclosed by the curve $y = \sqrt{x}$ and the lines $x = 1$ and $y = 0$.

2. Show that there is no vector field \mathbf{G} such that

$$\text{curl } \mathbf{G} = 2x \mathbf{i} + 3yz \mathbf{j} - xz^2 \mathbf{k}.$$

(Recall that $\text{curl } \mathbf{G}$ is the same as $\nabla \times \mathbf{G}$.)

3. The shape of a platform is given by

$$x^2 + y^2 \leq (2 - z)^2, \quad 0 \leq z \leq 1.$$

- (a) Describe this shape in cylindrical coordinates.
- (b) Hence find the volume of this platform.
- (c) Find the centroid of this platform. (By symmetry, it lies on the z -axis.)

4. Consider the plane Π with equation $2x + y - z = 11$.

- (a) Find the point Q of Π that is closest to the point $P = (2, 1, 0)$.
- (b) Find the distance PQ .
- (c) Verify that PQ and QR are perpendicular, where the point $R = (3, 6, 1)$ is known to lie on Π .

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5. The position of a particle at time t is given by

$$\mathbf{r}(t) = (\cos 2t, \sin 2t, t^2).$$

- (a) Find its *velocity* at time t .
- (b) Find its *acceleration* at time t .
- (c) Find its *speed* at time t (which is not the same as velocity).
- (d) Find the *unit tangent vector* \mathbf{T} at time t to the path traced out by the particle.

6. (a) State the Divergence Theorem. Explain briefly what each symbol in the theorem stands for. (You may assume all the differentiability you want.)

- (b) Use the Divergence Theorem to evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma,$$

where

$$\mathbf{F}(x, y, z) = xy \mathbf{i} + (y^2 + e^{xz}) \mathbf{j} + \sin(xy) \mathbf{k}$$

and S is the boundary surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$ and $y = 5$.

7. Let

$$\mathbf{F}(x, y, z) = (2x + y^2) \mathbf{i} + (2xy + 3y^2) \mathbf{j}.$$

- (a) Show that $\text{curl } \mathbf{F} = \mathbf{0}$.
- (b) Find a function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.
- (c) Use (b) to evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the arc of the curve $y = \sin^3 x$ from $(0, 0)$ to $(\pi/2, 1)$ in the xy -plane.

8. Given a function $g(x, y)$, convert to polar coordinates (r, θ) by setting $f(r, \theta) = g(r \cos \theta, r \sin \theta)$. (You may assume all the differentiability you want.)

- (a) Express the first-order partial derivatives of f in terms of g .
- (b) Express the second-order partial derivative $f_{\theta\theta}$ in terms of g . (Remember that g is a function of x and y .)

9. The Post Office will accept a parcel in the shape of a rectangular box provided the sum of its length and girth does not exceed 108 inches. (The *girth* is the circumference of a cross-section of the parcel perpendicular to its length.) What dimensions of such a parcel will maximize its volume?

10. (a) If f is a differentiable function whose value $f(x_0, y_0, z_0)$ is known, use differentials to *estimate* the value $f(x_0 + h_1, y_0 + h_2, z_0 + h_3)$ at a nearby point.

- (b) Use (a) to estimate

$$\sqrt[3]{2.99^2 + 3.04^2 + 3.06^2},$$

starting from $\sqrt[3]{3^2 + 3^2 + 3^2} = 3$.