# Final Examination

(Alternate Edition)

*Three hours. Closed book.* ***No notes. No calculators.***
*200 points, 20 per question.*
*Partial credit may be available, but only if you show your working.*
*Begin each of the ten questions on a new page and number it clearly in the margin.*
*If you use two books, label them "Book 1 of 2" and "Book 2 of 2". (If you use three books, . . . )*
*Make sure your T.A.'s name is on each book, as well as your name.*
*Do not use decimal evaluations of square roots, trigonometric functions and such.*
*All integrals have been well cooked to work out easily.*
*Use only the officially provided blue books.*

**1.** Show that there is no vector field **G** such that

$$\operatorname{curl}\mathbf{G} = 2x\,\mathbf{i} - 4yz\,\mathbf{j} + 3xz^2\,\mathbf{k}.$$

(Recall that curl **G** is the same as $\nabla \times \mathbf{G}$.)

**2.** The shape of a platform is given by

$$x^2 + y^2 \le (3 - z)^2, \qquad 0 \le z \le 2.$$

(a) Describe this shape in cylindrical coordinates.
(b) Hence find the volume of this platform.
(c) Find the centroid of this platform. (By symmetry, it lies on the $z$-axis.)

**3.** Consider the plane $\Pi$ with equation $2x + y - z = 17$.
(a) Find the point $Q$ of $\Pi$ that is closest to the point $P = (2, 1, 0)$.
(b) Find the distance $PQ$.
(c) Verify that $PQ$ and $QR$ are perpendicular, where the point $R = (5, 6, -1)$ is known to lie on $\Pi$.

**4.** The position of a particle at time $t$ is given by

$$\mathbf{r}(t) = (\cos 2t, -\sin 2t, t^2).$$

(a) Find its *velocity* at time $t$.
(b) Find its *acceleration* at time $t$.
(c) Find its *speed* at time $t$ (which is not the same as velocity).
(d) Find the *unit tangent vector* **T** at time $t$ to the path traced out by the particle.

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**5.** (a) State Green's Theorem.

(b) Use Green's Theorem to evaluate the contour integral

$$\int_C (1+y^6)dx + (x^2+e^y)dy$$

where $C$ denotes the boundary of the region enclosed by the curve $y = \sqrt{x}$ and the lines $x = 1$ and $y = 0$.

**6.** (a) If $f$ is a differentiable function whose value $f(x_0, y_0, z_0)$ is known, use differentials to *estimate* the value $f(x_0 + h_1, y_0 + h_2, z_0 + h_3)$ at a nearby point.

(b) Use (a) to estimate

$$\sqrt[3]{2.98^2 + 3.05^2 + 3.06^2},$$

starting from $\sqrt[3]{3^2 + 3^2 + 3^2} = 3$.

**7.** (a) State the Divergence Theorem. Explain briefly what each symbol in the theorem stands for. (You may assume all the differentiability you want.)

(b) Use the Divergence Theorem to evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma,$$

where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + (y^2 - e^{xz^2})\,\mathbf{j} + \cos(xy)\,\mathbf{k}$$

and $S$ is the boundary surface of the region $E$ bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$ and $y = 5$.

**8.** The Post Office will accept a parcel in the shape of a rectangular box provided the sum of its length and girth does not exceed 108 inches. (The *girth* is the distance around the perimeter of a cross-section of the parcel perpendicular to its length.) What dimensions of such a parcel will maximize its volume?

**9.** Let

$$\mathbf{F}(x, y, z) = (4x + y^2)\mathbf{i} + (2xy - 3y^2)\mathbf{j}.$$

(a) Show that $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

(b) Find a function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.

(c) Use (b) to evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C$ is the arc of the curve $y = \sin^3 x$ from $(0, 0)$ to $(\pi/2, 1)$ in the $xy$-plane.

**10.** Suppose $u = g(x, y)$, where the differentiable function $g$ is unknown. Put $x = r\cos\theta$ and $y = r\sin\theta$, so that $u = g(r\cos\theta, r\sin\theta) = f(r, \theta)$.

(a) Express the first-order partial derivatives of $f$ in terms of $g$.

(b) Express the second-order partial derivative $f_{\theta\theta}$ in terms of $g$. (Remember that $g$ is a function of $x$ and $y$, not $r$ and $\theta$.)

JMB, PG