First Examination

50 minutes. Closed book. No notes.
80 points, 20 per question.
Partial credit may be available, but only if you show your working.
Begin each of the four questions on a new page and number it clearly in the margin.
If you use two books, label them "Book 1 of 2" and "Book 2 of 2".
Make sure your T.A.'s name is on each book, as well as your name.
Calculators are allowed but are not recommended.
Do not numerically evaluate square roots and such.

1. (a) Show that

$$x^2 + y^2 + z^2 - 8x + 2y - 4z = 4$$

is the equation of a sphere. Find its center and radius.

(b) Find the equation of the plane that passes through the point $(6, 1, -1)$ and is normal to the vector $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

2. Consider the twisted cubic

$$\mathbf{r}(t) = \left(\frac{1}{3}t^3 + t\right)\mathbf{i} + \left(\frac{1}{3}t^3 - t\right)\mathbf{j} + (t^2 + 1)\mathbf{k}$$

(a) Show that the points $P(0, 0, 1)$ and $Q(12, 6, 10)$ lie on the curve, by finding the corresponding values of t.

(b) Calculate the arc length along the curve between the points P and Q. [No difficult integrals are needed.]

3. (a) Find a nonzero vector that is orthogonal to both of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, where $P = (1, 1, 0)$, $Q = (3, 2, 3)$, and $R = (2, 3, 1)$.

(b) Given the vector $\mathbf{d} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, decompose the vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ as the sum $\mathbf{b} + \mathbf{c}$ of two vectors, with $\mathbf{b}$ parallel to $\mathbf{d}$ and $\mathbf{c}$ orthogonal to $\mathbf{d}$.

4. The position in space of a particle at time t is given by

$$\mathbf{r}(t) = e^t\,\mathbf{i} + t^2\,\mathbf{j} + 2\sin t\,\mathbf{k}.$$

Find:

(a) The velocity vector at time $t = 0$;
(b) The speed at time $t = 0$;
(c) The acceleration vector at time $t = 0$;
(d) The parametric equation of the tangent line at time $t = 0$ to the curve traced out by the particle.