Mid-Term Examination

by J. Michael Boardman

50 minutes, 40 points, 10 points per question.
This is a closed book examination, with no notes or calculators allowed.
Partial credit may be available, but only if you show your working.
Begin each of the four questions on a new page and number it clearly in the margin.
In all questions, give reasons for your answers.
Use only the officially provided blue books.

1. For each of the following subsets of the plane $\mathbb{R}^2$, decide whether it is *open*, *closed*, or *neither*. In each case, give (at least) a short reason.

(a) The set of all points (x, y) such that x is an integer;
(b) The set of all points (x, y) such that $xy \geq 1$;
(c) The set of all points (x, y) such that $0 < x^2 + y^2 < 4$;
(d) The set of all points (x, y) such that $x + y$ is rational;
(e) The set of all points (x, y) such that $y = 0$ or $1/y$ is an integer.

2. (a) Given an infinite set X, consider the collection of all subsets U of X for which $X - U$ is infinite, empty, or the whole of X. Is this collection a topology on X?

(b) For each real a, put $U_a = (a, \infty) = \{x \in \mathbb{R} : a < x\}$. Do these sets form a basis of a topology on $\mathbb{R}$? If so, is it the standard topology?

3. Given a point x_0 in a metric space X, D, define the function $f\colon X \to \mathbb{R}$ by $f(x) = D(x_0, x)$. Prove that it satisfies the condition

$$|f(x) - f(y)| \leq D(x, y)$$

for all $x, y \in X$. Deduce that f is continuous. Is it possible to have $|f(x) - f(y)| < D(x, y)$?

4. (a) On the real line $\mathbb{R}$, define $D(x, y) = \sqrt{|y - x|}$. Is D a metric on $\mathbb{R}$?

(b) On the plane $\mathbb{R}^2$, define

$$D((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + (y_2 - y_1)^2.$$

Is D a metric on $\mathbb{R}^2$?

YOU MAY RETAIN THIS QUESTION SHEET