Mid-Term Examination

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50 minutes. 40 points, 10 per question.
THIS IS AN OPEN BOOK EXAMINATION.
Partial credit may be available, but only if you show your working.
Begin each of the four questions on a new page and number it clearly in the margin.
Don't forget to put your name on the cover of the book.
Use only the officially provided blue books.

1 Decide whether each of the following sets is finite, countably infinite, or uncountable. Give reasons.
(a) The set of all circles in the plane $\mathbb{R}^2$ that have integer radius;
(b) The set $\mathbb{Q} \times \mathbb{Z}$;
(c) The set of all order-preserving bijections $\mathbb{Z}_+ \to \mathbb{Z}_+$;
(d) The set of all order-preserving functions $\mathbb{Z}_+ \to \mathbb{Z}_+$;
(e) The set of all order-preserving bijections $\mathbb{Z} \to \mathbb{Z}$.

2 Define the relation $\sim$ on the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ by:

$$(x_1, y_1) \sim (x_2, y_2) \text{ if and only if } x_1 + y_1 = x_2 + y_2 + n \text{ for some integer } n.$$

Prove that $\sim$ is an equivalence relation. Describe a typical equivalence class for this relation.

3 On the plane $\mathbb{R}^2$, define

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_2 - x_1|, |x_2 + y_2 - x_1 - y_1|\}$$

Prove that d is a metric on $\mathbb{R}^2$. Sketch the ball $B((0,0), 2)$.

4 Give the real line $\mathbb{R}$ the standard ordering.
(a) Give an example of a non-empty subset of $\mathbb{R}$ that is bounded above but has no maximum (or greatest) element.
(b) Give an example of a proper subset of $\mathbb{R}$ that is not bounded above.
(c) Give an example of an ordered set X and a non-empty subset A that is bounded above but has no least upper bound.
(d) Give an example of a non-empty ordered set in which every element has an immediate predecessor and an immediate successor.
(e) Find *three* order relations on the set $\mathbb{Z}_+$ that have different order types.

YOU MAY RETAIN THIS QUESTION SHEET