Final Examination

Alternate Edition

Three hours. Closed book. ***No notes. No calculators.***

200 points, 20 per question.

Partial credit may be available, but only if you show your working.

Begin each of the ten questions on a new page and number it clearly in the margin.

If you use two books, label them "Book 1 of 2" and "Book 2 of 2". (If you use three books, . . .)

If you would like to have your grade posted on the Web page: *(a) Make a note of your secret number, which is on the inside front cover of Book 1.* *(b) When available, your grade will be listed by your secret number.* *If you prefer NOT to have your grade posted, cross out your secret number.*

Make sure your T.A.'s name is on each book, as well as your name.

Do not evaluate square roots, trigonometric functions and such.

All integrals have been well cooked to work out easily.

Use only the officially provided blue books.

1. Let $f(x, y, z) = x^3 - xz^2 + yz$, and P be the point $(1, 3, 1)$.

(a) For what value of c does P lie on the level surface $f(x, y, z) = c$?

(b) Find the equation, in terms of x, y, and z (i.e. without vectors), of the tangent plane at P to this level surface.

(c) Find a unit normal vector to this level surface at P.

2. (a) State the Divergence Theorem, and explain briefly what each symbol appearing in it means.

(b) Use the Divergence Theorem to calculate the total flux of the vector field

$$\mathbf{E}(x, y, z) = (x^3 - y^3)\mathbf{i} - xyz^2\,\mathbf{j} + (x^2z - 2x^2)\mathbf{k}$$

out of the box given by $0 \le x \le 1$, $0 \le y \le 3$, $0 \le z \le 2$.

3. Find all critical points of the function

$$g(x, y) = x^3 - 6xy + 3y^2 + 3$$

and classify them (as local maximum, local minimum, or saddle point).

4. Evaluate the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ of the vector field $\mathbf{F}(x, y, z) = 2z\,\mathbf{i} - y\,\mathbf{j} + x\,\mathbf{k}$ along the twisted cubic curve C given by $\mathbf{r}(t) = (t, t^2, t^3)$, where $0 \le t \le 2$.

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5. Consider the function $u(x, y, z) = \sqrt{x^3 + y^2 + z^2}$. [That's x^3, not x^2.]

(a) Find a formula for the differential du of u at the point $(2, 2, 2)$.

(b) Use (a) to find an *approximate* value of $u(2.02, 2.04, 1.98)$, starting from $u(2, 2, 2) = 4$.

6. Which of the following vector fields are gradient vector fields (i.e. of the form ∇f for some scalar field f)? For each one, either find such a scalar field f, or give a reason why no such f exists.

$$\mathbf{A}(x, y, z) = y^2 z\,\mathbf{i} + xz^2\,\mathbf{j} + x^2 y\,\mathbf{k}, \qquad \mathbf{B}(x, y, z) = x\,\mathbf{i} + y^2\,\mathbf{j} + z^3\,\mathbf{k},$$

$$\mathbf{C}(x, y, z) = (2xyz - 1)\mathbf{i} + (x^2 z - y)\mathbf{j} + x^2 y\,\mathbf{k}.$$

7. Find the volume of the solid enclosed by the two paraboloids $z = 2(x^2 + y^2)$ and $z = 2 + x^2 + y^2$. [*Hint:* Use cylindrical coordinates.]

8. Use Stokes' (or Green's) Theorem to find the circulation of the vector field

$$\mathbf{G}(x, y, z) = (e^x + x^2 y)\mathbf{i} + \sin y\,\mathbf{j} - e^{xyz}\,\mathbf{k}$$

around the piecewise-linear contour that starts at the origin $(0, 0, 0)$, goes to $(1, 0, 0)$, then to $(2, 1, 0)$, then to $(0, 1, 0)$, and back to $(0, 0, 0)$. [*Hint:* A sketch of the contour will help.]

9. Let Π be the plane $2x + y - z = 8$.

(a) Drop a perpendicular from the point $P = (7, 3, -3)$ to Π and find the foot Q of this perpendicular.

(b) Note that the point $R = (3, 7, 5)$ lies on Π. Find the angle between the line RP and the plane Π.

10. Consider the helix

$$\mathbf{r}(t) = (2t + 3\cos t)\mathbf{i} + (2t - 3\cos t)\mathbf{j} + 3\sqrt{2}\sin t\,\mathbf{k}.$$

(a) Find the unit tangent vector at a general point of the helix;

(b) Find the arc length of one turn of the helix, from $t = 0$ to $t = 2\pi$;

(c) Find the curvature of the helix.