Final Examination

Three hours, open book, no notes or calculators.
100 points, 10 per question.
Partial credit may be available, but only if you show your working.
Begin each of the ten questions on a new page and number it clearly in the margin.
If you use two books, label them "Book 1 of 2" and "Book 2 of 2". (If you use three books, ...)
Use only the officially provided blue books.

1. Let A and B be subsets of a space X. Is each of the following statements true or false? Give a proof or a counterexample.
(a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(b) $\text{Int}(A \cup B) = \text{Int}(A) \cup \text{Int}(B)$.

2. Given a space X, the *diagonal* Δ is the subset of $X \times X$ consisting of all points of the form (x, x), where $x \in X$. Show that X is Hausdorff if and only if Δ is closed in $X \times X$.

3. Let A and B be disjoint closed subsets of a compact metric space X. Show that there is a number $r > 0$ such that $d(x, y) \geq r$ for all $x \in A$ and $y \in B$. What happens if X is not compact?

4. Which of the following spaces are connected? Write down a separation of the space when there is one.
(a) The subspace $A = \{0\} \cup \{1/n : n \in \mathbb{Z}_+\}$ of $\mathbb{R}$;
(b) The subspace $B = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ of $\mathbb{R}^2$;
(c) The subspace $C = \{(x, y) \in \mathbb{R}^2 : x^2y = 1\}$ of $\mathbb{R}^2$;
(d) The space D consisting of $\mathbb{R}^2$ with three points removed.

5. Which of the following sets are countable? Give reasons.
(a) The set of all irrational (real) numbers;
(b) The set of all negative integers;
(c) The set of all linear functions of the form $ax + b$, with a and b rational;
(d) The set of all pairs of distinct odd integers;
(e) The set of all subsets of the real line $\mathbb{R}$.

6. Determine the fundamental group of each of the following subspaces of the plane $\mathbb{R}^2$. Give reasons.

(a) The plane $\mathbb{R}^2$ with the interval $[-1, 1] \times 0$ removed;
(b) The plane $\mathbb{R}^2$ with the half-line $[0, \infty) \times 0$ removed;
(c) The space of all points (x, y) with $y > 0$;
(d) The open 2-cell $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| < 1\}$.

7. (a) Suppose that d and d' are two metrics on the set X that satisfy the inequalities

$$md(x, y) \leq d'(x, y) \leq Md(x, y) \quad \text{for } x, y \in X,$$

where $m > 0$. Show that the two metrics are equivalent, i.e. define the same topology on X.

(b) Give an example of a set X with two metrics d and d' that do not satisfy the above inequalities but are nevertheless equivalent.

8. Find a sequence of continuous functions $f_n \colon \mathbb{R} \to \mathbb{R}$ and a function $f \colon \mathbb{R} \to \mathbb{R}$ that satisfy the following conditions:

(a) $f_n(x) \to f(x)$ as $n \to \infty$, for all $x \in \mathbb{R}$;
(b) f is *not* continuous.

9. Decide which of the following subspaces of $\mathbb{R}^3$ are compact. Give reasons.

(a) The unit disk $D^3 = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| \leq 1\}$;
(b) The unit sphere $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$;
(c) The unit cell $e^3 = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| < 1\}$;
(d) The subspace $\mathbb{Q}^3$ consisting of all points with rational coordinates.

10. Consider the ten digits

0 1 2 3 4 5 6 7 8 9

as subspaces of the plane $\mathbb{R}^2$ (treated as drawn using lines of zero width).

(a) Classify the digits up to homeomorphism.
(b) Classify the digits up to homotopy type.