Final Examination

by J. Michael Boardman

Three hours. 100 points, 10 per question.
This is an open book examination, but no notes or calculators are allowed.
Give reasons (which may be brief); they are worth about half the credit.
Partial credit may be available, but only if you show your working.
$\mathbb{R}^n$ *denotes euclidean* n*-space, with the standard topology.*
Subsets are assumed to be given the subspace topology.
Begin each of the ten questions on a new page and number it clearly in the margin.
If you use two books, label them "Book 1 of 2" and "Book 2 of 2". (If you use three books, ...)
Use only the officially provided blue books.

1. Decide, with reasons, whether each of the following subsets of $\mathbb{R}^2$ is *open, closed, both*, or *neither* [sketches may help]:
(i) $B =$ all points (x, y) that satisfy $x^2 + y^2 \leq 4$;
(ii) $C =$ all points (x, y) that have x rational and $|y| < 1$;
(iii) $E =$ all points (x, y) that satisfy $1 \leq x^2 + y^2 < 4$;
(iv) $F =$ the point $(2, 3)$ only;
(v) $G =$ all points (x, y) that satisfy $x^2 > 1$.

2. Which of the subspaces in Question 1 are *compact*? Give reasons.

3. Which of the subspaces in Question 1 are *connected*? Give reasons. Write down a separation (decomposition) when one exists.

4. Let X, D' and Y, D'' be metric spaces. Give a metric D for the product $X \times Y$ that induces the product topology on $X \times Y$ and *prove* that it does.

5. Decide whether each of the following sets is *finite, countably infinite* or *uncountable*. Give reasons.
(i) The unit circle, as a subset of $\mathbb{R}^2$;
(ii) The set of integer points (x, y) in $\mathbb{R}^2$ that lie within distance 1,000 of the origin;
(iii) The set of rational points (x, y) that satisfy $x^2 + y^2 \leq 1$;
(iv) The set of all sequences of rational numbers that converge to 0;
(v) The set of all polynomials in x, of any degree, with integer coefficients.

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6. Which of the following X, D are metric spaces? Give reasons.

(i) $X = \mathbb{R}^2$, with

$$D((x_1, y_1), (x_2, y_2)) = |x_2 - x_1|^4 + |y_2 - y_1|^4;$$

(ii) $X =$ all continuous real-valued functions on the interval $[0, 1]$, with

$$D(f, g) = \int_0^1 |g(x) - f(x)|\, dx;$$

(iii) $X =$ the set of all 50 states of the U.S.A., with $D(x, y) =$ distance in miles from the state capital of x to the state capital of y.

7. Find the fundamental group $\pi_1(X, x_0)$ (with your choice of basepoint x_0) of the following subspaces X of $\mathbb{R}^2$. [Sketches may help.]

(i) $X =$ all points (x, y) that satisfy $1 < x^2 + y^2 < 4$;
(ii) $X =$ all points (x, y) that satisfy $|x| < 1$;
(iii) $X =$ all points (x, y) that satisfy $4x^2 + y^2 = 1$;
(iv) $X =$ all points (x, y) that satisfy $4x^2 - y^2 = 1$.

8. Recall that by definition, a topology on X is a subset of the power set $\mathcal{P}(X)$ of X. Let τ_1 and τ_2 be two topologies on X.

(a) Is $\tau_1 \cup \tau_2$ a topology on X in general? Explain.
(b) Is $\tau_1 \cap \tau_2$ a topology on X in general? Explain.

9. Find two metric spaces that are homeomorphic, one of which is *complete* and the other *not* complete.

10. Given a subset A of a topological space X, compare the following five subsets of X. Which of them are subsets of which others?

(i) A itself;
(ii) $\operatorname{Int} A$, the *interior* of A;
(iii) $\operatorname{Cl} A$, the *closure* of A;
(iv) $\operatorname{Int}(\operatorname{Cl} A)$;
(v) $\operatorname{Cl}(\operatorname{Int} A)$.

YOU MAY RETAIN THIS QUESTION SHEET