Final Examination

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Three hours. Open book. No notes.

100 points, 10 per question.

Partial credit is available, but answers unsupported by good reasons will be worth little.

The spaces $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{Q}$ are always given the standard topology.

Begin each of the ten questions on a new page and number it clearly in the margin.

If you use two books, label them "Book 1 of 2" and "Book 2 of 2". (If you use three books, ...)

Use only the officially provided blue books.

1. Let $A \subset X$ and $B \subset Y$ be subspaces. In the space $X \times Y$, prove that $\overline{A \times B} = \overline{A} \times \overline{B}$.

2. Decide whether each of the following sets is finite, countably infinite, or uncountable:

The set A of all square-free positive integers. (An integer is *square-free* if it is not divisible by n^2 for any integer $n > 1$.)

The set B of all finite subsets of the real line $\mathbb{R}$.

The set C of all lines in the plane $\mathbb{R}^2$ that have rational slope.

The set D of all subsets of the set of all subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

3. Let A be a connected subspace of a space X.

(a) Is the interior $\operatorname{Int} A$ of A necessarily connected?

(b) Is the closure $\overline{A}$ of A necessarily connected?

(c) Is the boundary $\operatorname{Bd} A$ of A necessarily connected?

4. Let (X, d) be a metric space.

(a) If $x_0 \in X$, show that the function $f: X \to \mathbb{R}$ defined by $f(x) = d(x_0, x)$ is uniformly continuous.

(b) Given a nonempty subset A of X, show that the function $g: X \to \mathbb{R}$ defined by $g(x) = d(A, x)$ is uniformly continuous, where $d(A, x) = \inf_{y \in A} d(y, x)$.

5. Which of the following spaces are simply connected?

(a) The subspace $\mathbb{R}^2 - \{(0,0)\}$ of the plane $\mathbb{R}^2$;

(b) The real line $\mathbb{R}$;

(c) The ball $B(0, r)$ radius r in the plane $\mathbb{R}^2$.

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6. Let (X, d) and (Y, d) be complete metric spaces.

(a) Give a metric for the space $X \times Y$, and prove that it induces the product topology.

(b) Is your metric on $X \times Y$ complete?

7. Let A and B be compact subspaces of the Hausdorff space X. Are the spaces $A \cap B$ and $A \cup B$ necessarily compact?

8. Decide whether the following subsets of the plane $\mathbb{R}^2$ are open, closed, both, or neither. [Sketches may help.]

(a) The set of all (x, y) that satisfy $xy = \pm 1$;
(b) The set of all (x, y) for which x is rational and $|y| \leq 1$;
(c) The set of all (x, y) that satisfy $x^2 + y^2 = -2$;
(d) The set of all (x, y) in which x is an integer;
(e) The set of all (x, y) with $y^2 > 2$.

9. Let $p\colon X \to Y$ be a closed surjective continuous map, where X is normal. (p is a *closed* map if $p(C)$ is closed in Y whenever C is closed in X.) Prove that Y is normal.

10. Which of the following spaces are connected? Give a separation when there is one. [Sketches may help.]

(a) The set E of all points (x, y) in the plane $\mathbb{R}^2$ that satisfy $xy = 4$;
(b) The set F of all points (x, y) in the plane $\mathbb{R}^2$ that satisfy $4x^2 + y^2 = 1$;
(c) The set $\mathbb{Q}$ of all rational numbers;
(d) The set G of all points (x, y) in the plane $\mathbb{R}^2$ that satisfy $y = \pm x$.

YOU MAY RETAIN THIS QUESTION SHEET