Final Examination
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Three hours. Open book, no notes.
100 points, 10 per question (plus an extra-credit challenge question).
Partial credit may be available, but only if you show your working.
Begin each question on a new page and number it clearly in the margin.
If you use two books, label them “Book 1 of 2” and “Book 2 of 2”. (If you use
three books, . . .)
Use only the officially provided blue books.
The reals \( \mathbb{R} \) and \( \mathbb{R}^2 \) are understood to have the standard topology.
The integers are denoted by \( \mathbb{Z} \) and the positive integers by \( \mathbb{Z}^+ \).

1. Consider the collection of open balls \( B(x, 1/2^n) \) in \( \mathbb{R}^2 \), where \( x \in \mathbb{R}^2 \) has rational coordinates and \( n \) is a positive integer. Prove that this collection is a basis of the standard topology on \( \mathbb{R}^2 \). (A complete formal proof is required here.)

2. Determine whether each of the following subsets of \( \mathbb{R}^2 \) is open, closed, both, or neither. Give reasons. (Sketches may help.)
   \[ A = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y \neq 0\}; \]
   \[ B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 2\}; \]
   \[ C = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z}\}; \]
   \[ D = \{(x, y) \in \mathbb{R}^2 : x + y \in \mathbb{Q}\} \quad (\mathbb{Q} \text{ denotes the rationals}); \]
   \[ E = \{(x, y) \in \mathbb{R}^2 : x + y \leq 0\}. \]

3. Find which of the spaces in Question 2 are connected and which are path-connected. Give reasons. (Again, sketches may help.)

4. Let \( p : X \rightarrow Y \) be a surjective closed continuous map such that \( p^{-1}(y) \) is compact for each \( y \in Y \). If \( X \) is Hausdorff, show that \( Y \) is Hausdorff. (Recall that the map \( p \) is closed if \( p(C) \) is closed in \( Y \) whenever \( C \) is closed in \( X \).)

5. Is each of the following sets finite, countably infinite, or uncountable? Give reasons. If finite, find an upper bound for its cardinality.
   \( F \), the set of all equivalence relations on the set \( \mathbb{Z} \);
   \( G \), the set of all rationals of the form \( m/2^n \), where \( m \) and \( n \) are positive integers;
   \( H \), the set of all functions \( f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) that satisfy \( f(n) = n \) for all sufficiently large \( n \) (how large depends on \( f \));
   \( J \), the set of all topologies on a finite set with \( n \) elements;
   \( K \), the set of all (simple) order relations on the set \( \mathbb{Z} \).

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6. Which of the following spaces are *simply connected*? Give reasons.
   - $L$, the unit interval $[0, 1]$ in $\mathbb{R}$;
   - $M$, the subspace of all $x \in \mathbb{R}^2$ that satisfy $\|x\| \geq 2$;
   - $N$, the subspace of all $(x, y) \in \mathbb{R}^2$ that satisfy $y \neq 0$.

7. Let $X$ and $Y$ be metric spaces.
   - (a) Find a metric on $X \times Y$ that induces the product topology on $X \times Y$, and prove that it does.
   - (b) Does your metric make the projection map $\pi_1: X \times Y \to X$ uniformly continuous? Justify your answer.

8. Show that a connected normal space that contains more than one point is uncountable. [*Hint: The book’s definition (not universally accepted) of normal includes the condition that each singleton $\{x\}$ is closed.*]

9. Let $\mathcal{B}$ be a basis of the topological space $X$. Show that $X$ is compact if and only if each covering of $X$ by members of $\mathcal{B}$ has a finite subcovering. (Again, a real formal proof is required.)

10. Find the *interior*, *boundary* and *closure* of each of the following subsets:
    - $P = \{x \in \mathbb{R}^2 : 1 \leq \|x\| < 2\}$, a subset of $\mathbb{R}^2$;
    - $Q$, the set of all irrational points in the unit interval $[0, 1]$, considered as a subset of $\mathbb{R}$.

11. (extra-credit challenge question) Given (simply) ordered sets $X$ and $Y$, find necessary and sufficient conditions for the dictionary order topology on $X \times Y$ to coincide with the product topology. (Assume that $X$ and $Y$ both contain at least three points.)

YOU MAY RETAIN THIS QUESTION SHEET