Final Examination

by J. Michael Boardman, Department of Mathematics

Three hours. Open book, no notes.
100 points, 10 per question (plus an extra-credit challenge question).
Partial credit may be available, but only if you show your working.
Begin each question on a new page and number it clearly in the margin.
If you use two books, label them "Book 1 of 2" and "Book 2 of 2". (If you use three books, ...)
Use only the officially provided blue books.
The reals $\mathbb{R}$ and $\mathbb{R}^2$ are understood to have the standard topology.
The integers are denoted by $\mathbb{Z}$ and the positive integers by $\mathbb{Z}_+$.

1. Consider the collection of open balls $B(x, 1/2^n)$ in $\mathbb{R}^2$, where $x \in \mathbb{R}^2$ has rational coordinates and n is a positive integer. Prove that this collection is a basis of the standard topology on $\mathbb{R}^2$. (A complete formal proof is required here.)

2. Determine whether each of the following subsets of $\mathbb{R}^2$ is *open*, *closed*, *both*, or *neither*. Give reasons. (Sketches may help.)
$A = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y \neq 0\}$;
$B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 2\}$;
$C = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z}\}$;
$D = \{(x, y) \in \mathbb{R}^2 : x + y \in \mathbb{Q}\}$ ($\mathbb{Q}$ denotes the rationals);
$E = \{(x, y) \in \mathbb{R}^2 : x + y \leq 0\}$.

3. Find which of the spaces in Question 2 are *connected* and which are *path-connected*. Give reasons. (Again, sketches may help.)

4. Let $p: X \to Y$ be a surjective closed continuous map such that $p^{-1}(y)$ is compact for each $y \in Y$. If X is Hausdorff, show that Y is Hausdorff. (Recall that the map p is *closed* if $p(C)$ is closed in Y whenever C is closed in X.)

5. Is each of the following sets *finite*, *countably infinite*, or *uncountable*? Give reasons. If finite, find an upper bound for its cardinality.
F, the set of all equivalence relations on the set $\mathbb{Z}$;
G, the set of all rationals of the form $m/2^n$, where m and n are positive integers;
H, the set of all functions $f: \mathbb{Z}_+ \to \mathbb{Z}_+$ that satisfy $f(n) = n$ for all sufficiently large n (how large depends on f);
J, the set of all topologies on a finite set with n elements;
K, the set of all (simple) order relations on the set $\mathbb{Z}$.

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6. Which of the following spaces are *simply connected*? Give reasons.
L, the unit interval $[0, 1]$ in $\mathbb{R}$;
M, the subspace of all $x \in \mathbb{R}^2$ that satisfy $\|x\| \geq 2$;
N, the subspace of all $(x, y) \in \mathbb{R}^2$ that satisfy $y \neq 0$.

7. Let X and Y be metric spaces.
(a) Find a metric on $X \times Y$ that induces the product topology on $X \times Y$, and *prove* that it does.
(b) Does your metric make the projection map $\pi_1 \colon X \times Y \to X$ *uniformly* continuouos? Justify your answer.

8. Show that a connected normal space that contains more than one point is uncountable. [*Hint:* The book's definition (not universally accepted) of *normal* includes the condition that each singleton $\{x\}$ is closed.]

9. Let $\mathcal{B}$ be a basis of the topological space X. Show that X is compact if and only if each covering of X by members of $\mathcal{B}$ has a finite subcovering. (Again, a real formal proof is required.)

10. Find the *interior*, *boundary* and *closure* of each of the following subsets:
$P = \{x \in \mathbb{R}^2 : 1 \leq \|x\| < 2\}$, a subset of $\mathbb{R}^2$;
Q, the set of all irrational points in the unit interval $[0, 1]$, considered as a subset of $\mathbb{R}$.

11. (extra-credit challenge question) Given (simply) ordered sets X and Y, find necessary and sufficient conditions for the dictionary order topology on $X \times Y$ to coincide with the product topology. (Assume that X and Y both contain at least three points.)

YOU MAY RETAIN THIS QUESTION SHEET