Final Examination

Three hours.
Closed book, but ONE page (one side) of formulae is allowed.
All integrals have been well cooked to calculate out easily.
Calculators are allowed but are not useful or recommended.
Do not evaluate square roots, π, and such.
200 points, 20 per question.
Partial credit may be available, but only if you show your working.
Begin each of the ten questions on a new page and number it clearly in the margin.
If you use two books, label them "Book 1 of 2" and "Book 2 of 2". (If you use three books, ...)
Make sure your T.A.'s name is on each book, as well as your name.
Use only the officially provided blue books.

1. Consider the vector field

$$\mathbf{F}(\mathbf{r}) = e^y\mathbf{i} + (xe^y - e^y + \frac{1}{3}z^3)\mathbf{j} + (yz^2 - z^2)\mathbf{k}.$$

(a) Show that $\nabla \times \mathbf{F} = \mathbf{0}$.
(b) Find a scalar field $g(\mathbf{r})$ such that $\mathbf{F} = \nabla g$.

2. Use Stokes's Theorem to evaluate the line integral $\int_C \mathbf{G}(\mathbf{r}){\cdot}d\mathbf{r}$, where C is the triangular contour that goes from $(2,0,0)$ to $(0,2,0)$ to $(0,0,2)$ and back to $(2,0,0)$, and $\mathbf{G}$ is the vector field

$$\mathbf{G}(\mathbf{r}) = (x^2 + y)\mathbf{i} + 4(x - y)\mathbf{j} + z^3\mathbf{k}.$$

3. (a) Find an equation of the plane that passes through the point $(1,2,3)$ and is normal to the vector $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.
(b) Find the (shortest) distance from the point $(3,4,5)$ to this plane.

4. A solid cylinder with radius 2 has the z-axis as its axis.
(a) Use cylindrical coordinates to describe the region that is cut out from this cylinder by the paraboloid $z = x^2 + y^2$ and the plane $z = x + 7$.
(b) Hence find the volume of this region.

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5. Use Gauss's Divergence Theorem to calculate the flux of the vector field

$$\mathbf{v}(\mathbf{r}) = x^2 y^2 \mathbf{i} + xz^3 \mathbf{j} + y^4 \mathbf{k}$$

out of the cube given by $0 \le x \le 2$, $0 \le y \le 2$, and $0 \le z \le 2$.

6. Treat the earth as a sphere of radius R, centered at the origin, with the north pole on the positive z-axis.

(a) Describe in spherical coordinates the region of the earth's surface that lies north of the latitude line 45 degrees North.

(b) Find the area of this region.

7. Find the maximum and minimum values attained by the function

$$f(x, y) = x^2 - 5x + 2y^2 - 5y + 6$$

anywhere on the triangle whose vertices are $(0, 0)$, $(2, 0)$, and $(0, 4)$. Also, find the points where these values are attained.

8. Reverse the order of integration in the integral

$$I = \int_0^2 \left\{ \int_{-x}^{x^3} x^2 y\, dy \right\} dx$$

so as to integrate first with respect to x.

9. The water temperature T at the point (x, y, z) is given by $T = x^2 e^y (1 + \sin 2z)$. A fish has reached the point $(1, 0, 0)$.

(a) What rate of increase of temperature does the fish experience if it starts swimming with the velocity vector $5\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$?

(b) In which direction should the fish swim to maximize the rate of increase of temperature? [The answer should be a unit vector.]

(c) In which direction should the fish swim to maximize the rate of decrease of temperature? [The answer should be a unit vector.]

10. The position of a particle at time t is given by $\mathbf{r}(t) = t\mathbf{i} - t^2\mathbf{j} + (2/3)t^3\mathbf{k}$.

(a) Find the velocity of the particle at time t.

(b) Find the acceleration of the particle at time t.

(c) Find the speed of the particle at time t.

(d) Find the distance (arc length) traveled between the times $t = 0$ and $t = 3$.