110.202 CALCULUS III Spring 1998

Final Examination (Alternate B)

Three hours.
Closed book, but ONE page (one side) of formulae is allowed.
200 points, 20 per question.
Partial credit may be available, but only if you show your working.
Calculators are allowed but are not useful or recommended.
Do not evaluate square roots, π, and such.
Integrals have been well cooked to calculate out easily.
In cartesian coordinates (x, y, z), $\mathbf{r}$ denotes the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
Begin each of the ten questions on a new page and number it clearly in the margin.
If you use two books, label them "Book 1 of 2" and "Book 2 of 2". (If you use three books, ...)
Make sure your name is on each book.
Use only the officially provided blue books.

1. The position of a roller coaster (considered to be a point) at time t is given by $\mathbf{r}(t) = 2t\mathbf{i} - t^2\mathbf{j} + (t^3/3)\mathbf{k}$.
(a) Find the velocity of the roller coaster at time t.
(b) Find the acceleration of the roller coaster at time t.
(c) Find the speed of the roller coaster at time t.
(d) Find the distance (along the track) traveled between the times $t = 0$ and $t = 2$.

2. A storage shed in the shape of a rectangular box, with four walls and a flat roof, is to be constructed out of sheet metal. The ground will serve as the floor. However, half of one long wall will be omitted, to serve as an entrance. The other three walls will each have an opening of 2 square meters for a window. If the capacity is 24 cubic meters, what dimensions of the box (length, width, and height) will require the least material?

3. Use Gauss's Divergence Theorem to calculate the total flux out of the cube Ω, given by $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$, of the vector field

$$\mathbf{v}(\mathbf{r}) = 2xy\mathbf{i} + (y - y^2)\mathbf{j} + (x^2y + z)\mathbf{k}.$$

4. A solid ball radius a is centered at the origin. A cylindrical hole radius b (where $0 < b < a$) is drilled through the center of the ball, with the axis of the hole along the z-axis, to leave a bead.
(a) Describe the bead in cylindrical coordinates (r, θ, z).
(b) Hence find the volume of the bead.
(c) What happens as b approaches a, or as b approaches 0?

OVER →

5. (a) Find all distinct third-order partial derivatives of the function $f(x,y) = x^3e^{2y}$.

(b) List all the distinct third-order partial derivatives of a general function of three variables, $g(x,y,z)$. How many are there? (You may assume g has continuous partial derivatives of all orders.)

6. Consider the vector field

$$\mathbf{F}(\mathbf{r}) = (e^y - xz^2)\mathbf{i} + xe^y\mathbf{j} + (z^3 - x^2z)\mathbf{k}.$$

(a) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$.

(b) Find a scalar field $h(\mathbf{r})$ such that $\mathbf{F} = \nabla h$.

7. (a) Find an equation of the plane Π that is parallel to the plane $2x - 2y + z = 1$ and passes through the point $(5,3,1)$.

(b) Find the (shortest) distance from the origin to the plane Π.

8. Consider the vector field

$$\mathbf{G}(\mathbf{r}) = y^2z\mathbf{i} + (x^2y + z^2 - 3z)\mathbf{j} + (2yz + e^z)\mathbf{k}$$

(a) Use Stokes's Theorem to express the line integral $\int_C \mathbf{G}(\mathbf{r}) \bullet d\mathbf{r}$ as a surface integral, where C denotes the piecewise linear (square) contour that goes from the origin to $(0,2,0)$, then to $(0,2,2)$, then to $(0,0,2)$, and back to the origin.

(b) Hence evaluate the line integral. [HINT: Do not evaluate the line integral directly unless you have lots of time and want to check your answer.]

9. Consider the iterated integral

$$I = \int_{-2}^{2} \left\{ \int_{y^2}^{4} \sqrt{x}y^2e^{x^3}\,dx \right\} dy.$$

(a) Sketch the region of integration.

(b) Reverse the order of integration, by expressing I as an iterated integral with y integrated first.

(c) Hence evaluate I. [WARNING: Do not attempt to evaluate I directly!]

10. The water temperature T at the point (x,y,z) is given by

$$T(x,y,z) = x^2 + x - xy - yz + z^2.$$

A fish has reached the point $P = (1,1,1)$.

(a) Find the gradient vector of T at the point P.

(b) What rate of increase of temperature does the fish experience at P if it swims with the velocity vector $5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$?

(c) In which direction from P should the fish swim in order to maximize the rate of increase of temperature? [The answer should be a unit vector.] What is this maximum rate of increase, if the fish has a maximum speed of V?

(d) In which direction from P should the fish swim in order to maximize the rate of *decrease* of temperature? [The answer should be a unit vector.]