Solutions to First Examination

10:00 a.m. Edition

1. (a) The vector $\mathbf{d} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ is parallel to the plane p. Since the points $(3, 2, 5)$ and $(1, 1, 1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ lie in the plane, the difference vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ is parallel to the plane. Therefore to find a normal vector to the plane, take the cross product

$$\mathbf{N} = \mathbf{d} \times (2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}.$$

(Any nonzero multiple of this will do.) (b) Then the equation of the plane p is $(\mathbf{r} - \mathbf{i} - \mathbf{j} - \mathbf{k}) \bullet \mathbf{N} = 0$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, which expands to $5(x-1) + 2(y-1) - 3(z-1) = 0$, which simplifies to $5x + 2y - 3z = 4$. (Again, any nonzero multiple of this will do.)

2. P is the point $(1, 1, 0) = \mathbf{i} + \mathbf{j}$. For (a), differentiate:

$$\mathbf{r}'(t) = 2t\mathbf{i} + (3t^2 - 2)\mathbf{j} + (4t^3 - 2)\mathbf{k}$$

and set $t = 1$, to get $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ as a tangent vector. Then $\|\mathbf{v}\| = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$, so that the unit tangent vector is $\mathbf{T} = \mathbf{v}/3 = (2/3)\mathbf{i} + (1/3)\mathbf{j} + (2/3)\mathbf{k}$. Then for (b), the parametric equation of the tangent line at P is

$$\mathbf{R}(u) = \mathbf{i} + \mathbf{j} + u\mathbf{v} = \mathbf{i} + \mathbf{j} + u(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}).$$

(Or we could replace $\mathbf{v}$ by $\mathbf{T}$ here.)

3. (a) First compute the derivative

$$\mathbf{r}'(t) = 2t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{k}.$$

Its norm was cooked to work out nicely,

$$\|\mathbf{r}'(t)\| = \sqrt{4t^4 + 4t^2 + 1} = 2t^2 + 1.$$

The relevant values of t are $-1 \le t \le 2$ (look at the $\mathbf{k}$-component of $\mathbf{r}(t)$). Then the arc length is

$$\int_{-1}^{2} (2t^2 + 1)dt = [2t^3/3 + t]_{-1}^{2} = 16/3 + 2 - (-2/3 - 1) = 9.$$

(b) The direct distance QR is $\sqrt{6^2 + 3^2 + 3^2} = \sqrt{54}$.

4. (a) First compute $\|\mathbf{a}\| = \sqrt{4 + 1 + 1} = \sqrt{6}$ and $\|\mathbf{b}\| = \sqrt{1 + 1 + 1} = \sqrt{3}$. The dot product is $\mathbf{a} \bullet \mathbf{b} = 2 - 1 + 1 = 2$. If the angle between the vectors is θ,

$$\cos\theta = (\mathbf{a} \bullet \mathbf{b})/\|\mathbf{a}\|\|\mathbf{b}\| = 2/\sqrt{3}\sqrt{6} = \sqrt{2}/3.$$

So $\theta = \cos^{-1}(\sqrt{2}/3)$ (which does not simplify). In (b), the angle is simply $\pi - \theta = \pi - \cos^{-1}(\sqrt{2}/3) = \cos^{-1}(-\sqrt{2}/3)$.

JMB