110.202 CALCULUS III 23 February 2000

Solutions to First Examination

11:00 a.m. Edition

11. The relevant values of t are $0 \le t \le 3$ (look at the $\mathbf{k}$-component of $\mathbf{r}(t)$). Differentiate,

$$\mathbf{r}'(t) = t^2\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}.$$

Its norm has been cooked to work out nicely,

$$\|\mathbf{r}'(t)\| = \sqrt{t^4 + 4t^2 + 4} = t^2 + 2.$$

Then the arc length is

$$\int_0^3 (t^2 + 2)dt = [t^3/3 + 2t]_0^3 = 9 + 6 = 15.$$

12. Call these points P_1, P_2 and P_3. Then the difference vectors $\overrightarrow{P_1P_2} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\overrightarrow{P_1P_3} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ (and $\overrightarrow{P_2P_3} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$) are parallel to the plane p. (a) A normal vector to the plane is

$$\mathbf{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 2 & 3 & 2 \end{vmatrix} = -7\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

(b) The equation of p is $\mathbf{N} \bullet (\mathbf{r} - \overrightarrow{OP_1}) = 0$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, which reduces to $-7x + 4y + z = 7$. (Check that the constant term makes P_1 lie on p.) (Any nonzero multiple of this equation or $\mathbf{N}$ will do.)

13. (a) Compute the norms $\|\mathbf{a}\| = \sqrt{1 + 4 + 4} = 3$ and $\|\mathbf{b}\| = \sqrt{1 + 1 + 9} = \sqrt{11}$. The dot product is $\mathbf{a} \bullet \mathbf{b} = 1 + 2 + 6 = 9$. So if the angle is θ,

$$\cos\theta = \mathbf{a} \bullet \mathbf{b}/\|\mathbf{a}\|\|\mathbf{b}\| = 9/3\sqrt{11} = 3/\sqrt{11}$$

and $\theta = \cos^{-1}(3/\sqrt{11})$. (b) Compute $2\mathbf{a} - \mathbf{b} = 2(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Its norm is $\sqrt{1 + 9 + 1} = \sqrt{11}$.

14. At P, $t = 1$ (from the $\mathbf{k}$-component). (a) Differentiate, to get

$$\mathbf{r}'(t) = (2t - 1)\mathbf{i} + (-3t^2 + 1)\mathbf{j} + 2\mathbf{k}.$$

Then $\mathbf{r}'(1) = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, with norm $\|\mathbf{r}'(1)\| = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$. So the unit tangent vector at P is $\mathbf{T} = \mathbf{r}'(1)/3 = (1/3)\mathbf{i} - (2/3)\mathbf{j} + (2/3)\mathbf{k}$. (b) The tangent line to E at $P = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is

$$\mathbf{R}(u) = \mathbf{i} + \mathbf{j} + \mathbf{k} + u\mathbf{T} = \mathbf{i} + \mathbf{j} + \mathbf{k} + u((1/3)\mathbf{i} - (2/3)\mathbf{j} + (2/3)\mathbf{k}),$$

or we can replace $\mathbf{T}$ here by $\mathbf{r}'(1)$.

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