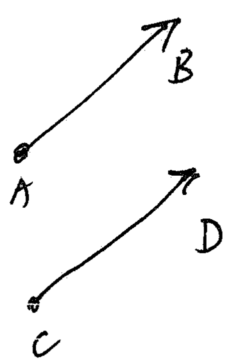
Chapter 1

1.1 vectors

points and coordinates are used to describe position (location) while vectors are used to describe displacements which has a magnitude (size) and direction (e.g. velocity, force, ...)

Informally a vector is an arrow joining two points A, B, notation $\overrightarrow{AB}$



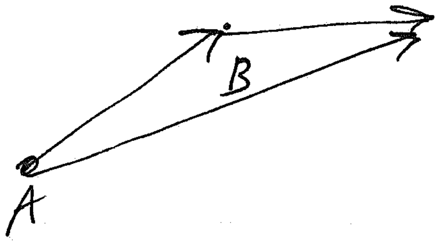
$\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$ i.e represent the same vector if $\overrightarrow{AB}, \overrightarrow{CD}$ are parallel with the same length and orientation.

Given a point $X = (x_1, \dots, x_n)$ (usually $n = 2, 3$) in a coordinate system with origin $O = (0, \dots, 0)$, we call $\vec{OX}$ the *position vector* of the point X.

$$\vec{v} + \vec{w}$$

vector addition $\quad \vec{AB} + \vec{BC} = \vec{AC}$

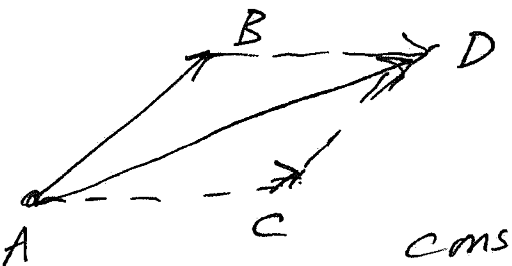
method 1



represent $\vec{v}$ by $\vec{AB}$

represent $\vec{w}$ by $\vec{BC}$

method 2

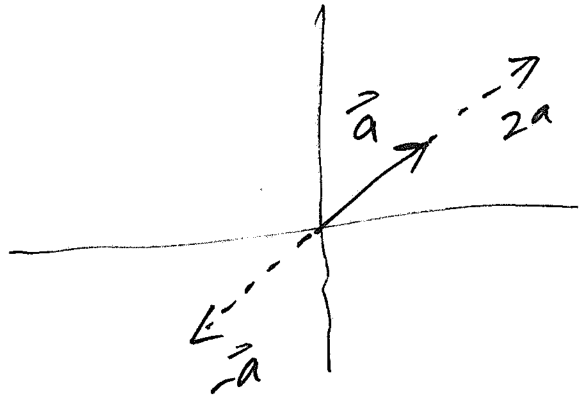


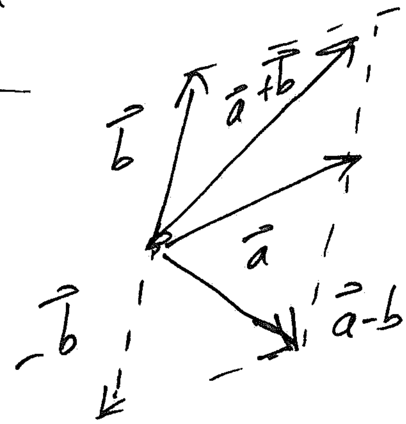
represent w by $\vec{AC}$

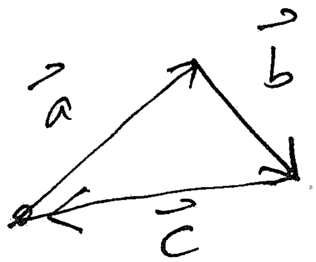
construct parallelogram

so diagonal $\vec{AD}$ then represents $\vec{v} + \vec{w}$

multiplication by a scalar







$$\vec{a}+\vec{b}+\vec{c}=\vec{0}$$

$$\vec{a}+\vec{b}=\vec{b}+\vec{a}$$

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$$

$$t(\vec{a}+\vec{b})=t\vec{a}+t\vec{b}$$

$$t(s\vec{a})=(ts)\vec{a}$$

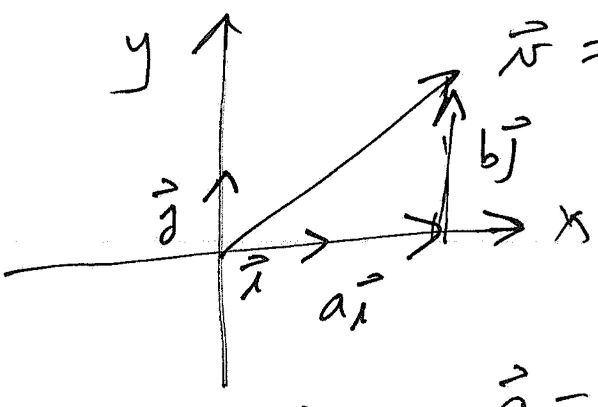
$$(t+s)\vec{a}=t\vec{a}+s\vec{a}$$

Component representation of vectors

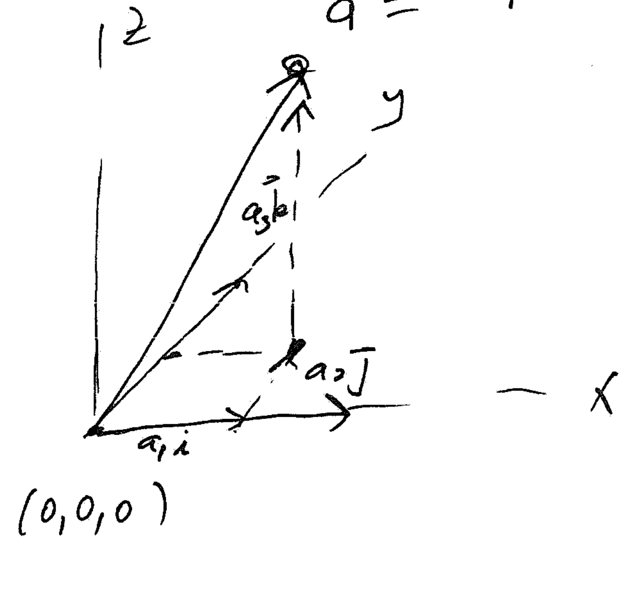
Let $\vec{i}, \vec{j}, \vec{k}$ be unit vectors in the direction of the (positive) x, y, z axes

(modern terminology for coord. system $(x_1, \ldots, x_n)$ $\quad \vec{e}_1, \ldots, \vec{e}_n$)

$\vec{v} = a\vec{i} + b\vec{j}$



$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$



For $\vec{a} = a_1 \vec{e}_1 + a_2 \vec{e}_2$

$\|\vec{a}\| =$ length $\vec{a}$ $= \sqrt{a_1^2 + a_2^2}$

by Pythagorean thm.

Similarly $\vec{a} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + a_3 \vec{e}_3$

$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

<u>dot product</u> and <u>projection</u>

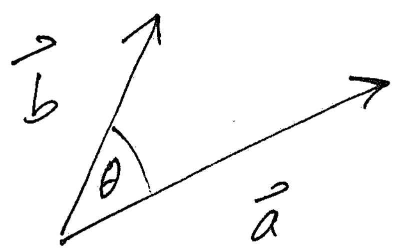
given vectors $\vec{a}$ and $\vec{b}$

in $\mathbb{R}^n$ $(n = 2, 3, \ldots)$

define the <u>dot product</u>

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

where θ $(0 \le \theta \le \pi)$
is the <u>angle</u> between $\vec{a}$ + $\vec{b}$

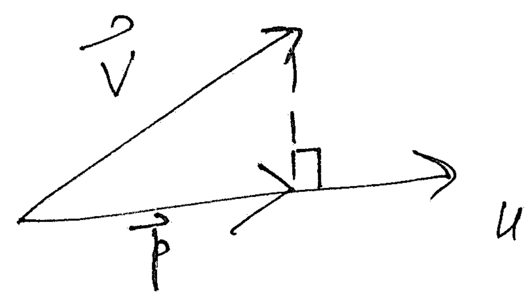


<u>Note</u> Any two non-parallel vectors (linearly independent) span a 2-dim'l plane

$$\text{span}(\vec{u}, \vec{v}) = \{ t_1 \vec{u} + t_2 \vec{v},\ t_1, t_2 \in \mathbb{R} \}$$

(6.5)

(orthogonal) projection



$$\vec{p} = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|^2} \vec{u}$$

"vector projection"

$\vec{p} = \mathrm{Proj}_{\vec{u}} \vec{v}$

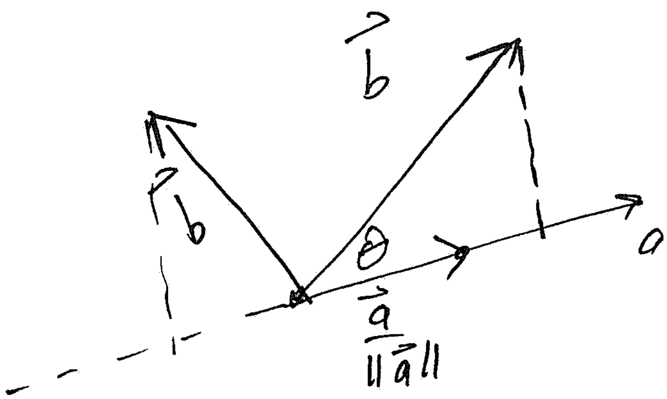
$$|\vec{p}| = \frac{|\vec{v} \cdot \vec{u}|}{|\vec{u}|}$$

"scalar" projection

$$\frac{\vec{v} \cdot \vec{u}}{|\vec{u}|}$$

= signed length.

If $\|\vec{a}\| \neq 0$, $\dfrac{\vec{a}}{\|\vec{a}\|}$ is a unit vector in the direction of $\vec{a}$



$$\vec{b} \cdot \frac{\vec{a}}{\|\vec{a}\|} = |\vec{b}| \cos\theta$$

(positive if $0 \le \theta < \pi/2$, negative if $\pi/2 < \theta \le \pi$)

represents the (signed) length of "The projection of $\vec{b}$ onto $\vec{a}$"

Note $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

so if $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$

$\vec{w} = w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k}$

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2$

since $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$

$\vec{i}\cdot\vec{j} = \vec{i}\cdot k = 0$ etc

and we can "distribute the dot product." because the dot product is a linear operation

$(\vec{u}+\vec{v})\cdot\vec{w} = \vec{w}\cdot(\vec{u}+\vec{v}) =$ etc

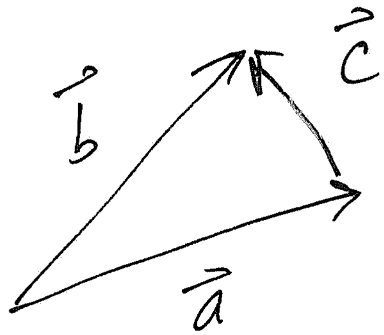
$\vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$

Note $\vec{v} \perp \vec{w}$ if and only if $\vec{v}\cdot\vec{w} = 0$

$\vec{v}\cdot\vec{w} > 0 \longleftrightarrow \theta < \pi/2$

$\vec{v}\cdot\vec{w} < 0 \longleftrightarrow \theta > \pi/2$

The dot product is equivalent to the law of cosines



$$\vec{a} + \vec{c} = \vec{b}$$

$$\vec{c} = \vec{b} - \vec{a}$$

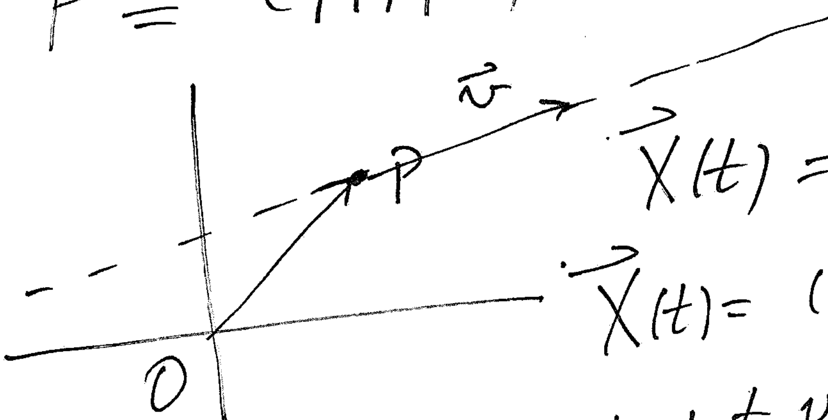
$$|\vec{c}|^2 = (\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a}) =$$

$$|\vec{b}|^2 + |\vec{a}|^2 - 2\|\vec{a}\|\|\vec{b}\| \cos\theta$$

"parametric"

equation of a line through a pt P in the direction of a vector $\vec{v}$

$n=2$ $\quad P = (p_1, p_2) \qquad \vec{v} = v_1 \vec{i} + v_2 \vec{j}$



$$\vec{X}(t) = \vec{OP} + t\,\vec{v}$$

$$\vec{X}(t) = (x(t), y(t))$$

$$\begin{cases} x(t) = p_1 + t\,v_1 \\ y(t) = p_2 + t\,v_2 \end{cases}$$

If both $v_1, v_2 \neq 0$ sometimes you see this written as

$$\frac{x - p_1}{v_1} = \frac{y - p_2}{v_2}$$

with the parameter t removed!

Similarly in $\mathbb{R}^3$,

$P = (p_1, p_2, p_3)$ $\quad \vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$

$$\vec{X}(t) = \vec{OP} + t\,\vec{v} = \vec{P} + t\,\vec{v}$$

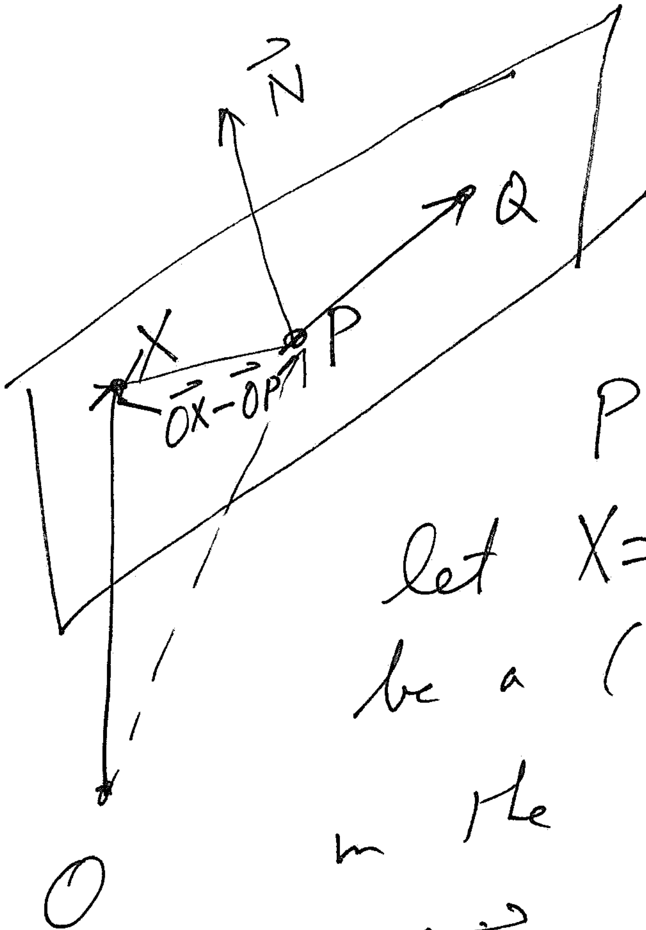
$$x(t) = p_1 + t v_1$$
$$y(t) = p_2 + t v_2$$
$$z(t) = p_3 + t v_3$$

The equation of a plane in $\mathbb{R}^3$

through a point P perpendicular

to a (normal) vector $\vec{N}$:

$$= \{ Q \in \mathbb{R}^3 : \vec{PQ} \cdot \vec{N} = 0 \}$$



$$\vec{N} = n_1 \vec{i} + n_2 \vec{j} + n_3 \vec{k}$$

$$P = (p_1, p_2, p_3)$$

let $X = (x, y, z)$ be a (variable) point in the plane

Then $(\vec{OX} - \vec{OP}) \cdot \vec{N} = 0$

$$\boxed{n_1(x - p_1) + n_2(y - p_2) + n_3(z - p_3) = 0}$$

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The cross product

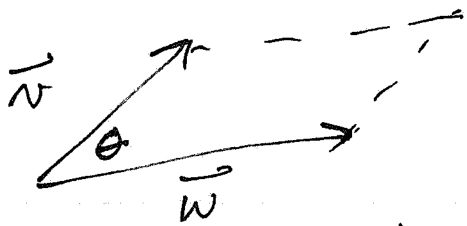
Given two vectors $\vec{v}, \vec{w}$

we define the cross-product $\vec{v} \times \vec{w}$ to be a vector

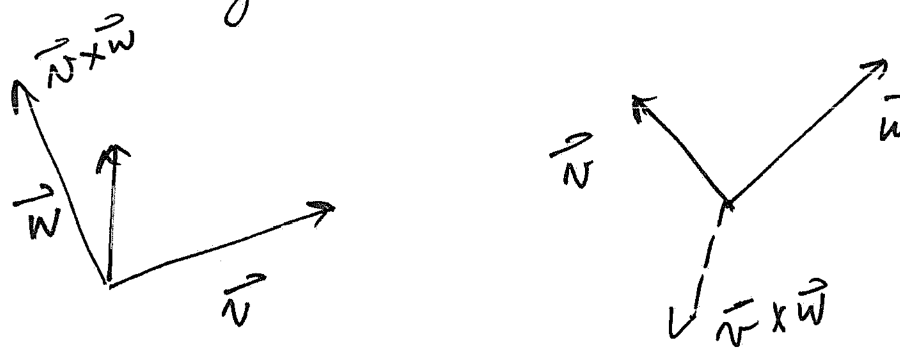
s. that i) $\vec{v} \times \vec{w} \perp \operatorname{span}\{\vec{v}, \vec{w}\}$

ii) $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \sin\theta$

= area of the parallelogram spanned by $\vec{v}$ & $\vec{w}$



and iii) $\vec{v} \times \vec{w}$ satisfies the "right hand rule"



If $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$

$\vec{w} = w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k}$

then using $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$

$$\vec{i} \times \vec{j} = -\vec{j} \times \vec{i} = \vec{k}$$

$$\vec{j} \times \vec{k} = -\vec{k} \times \vec{j} = \vec{i}$$

$$\vec{k} \times \vec{i} = -\vec{i} \times \vec{k} = \vec{j}$$

we find $\vec{v} \times \vec{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$

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Introduce 2×2, 3×3 determinants

2×2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

and its determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

Similarly a 3×3 matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and its determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(One can also "expand by cofactors" along first column

Formally $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$

$\vec{w} = w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & \vec{v}_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Parametric equation of a plane

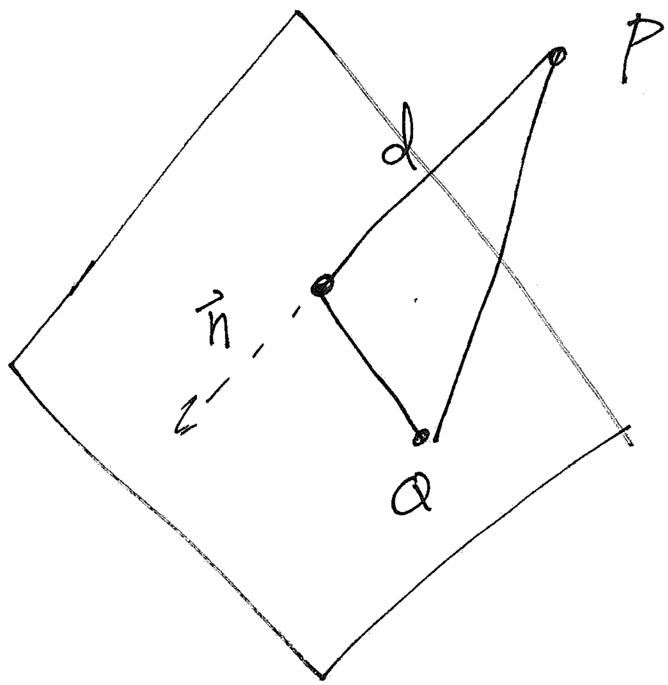
A plane Π in R^3 through a point P is determined by two vectors $\vec{u}, \vec{v}$ that are parallel to the plane. That is,

$$\Pi = \{ (x,y,z) \in \mathbb{R}^3 : \vec{X}(s,t) = (x,y,z) = \vec{P} + s\vec{u} + t\vec{v} \}$$

$$= \left\{ (x,y,z) \in R^3 : \begin{cases} x = p_1 + s u_1 + t v_1 \\ y = p_2 + s u_2 + t v_2 \\ z = p_3 + s u_3 + t v_3 \end{cases} \right\}$$

so $\Pi = \vec{P} + \operatorname{span}\{\vec{u}, \vec{v}\}$

Example 1. Find the distance between $P=(1,0,-1)$ and the plane $5x+4y+3z=1$



We need an additional point Q in the plane. For example take $x=y=0$ $z=1/3$

$$\text{So} \quad Q=(0,0,\tfrac{1}{3})$$

Take $\vec{n} = 5\vec{i}+4\vec{j}+3\vec{k}$

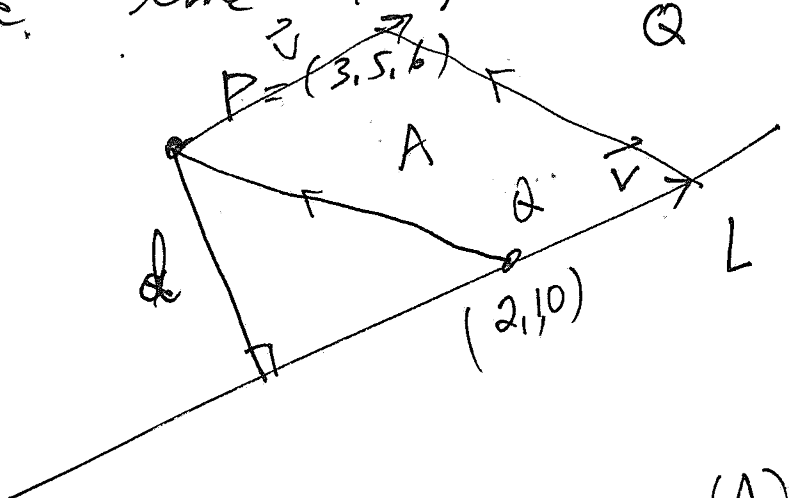
Then $d = \left| \text{Pr}_{\vec{n}} \overrightarrow{QP} \right|$

$$= \left| \frac{\overrightarrow{QP} \cdot \vec{n}}{|\vec{n}|} \right|$$

$$= \left| \frac{\left(\vec{\imath} - 4/3\,\vec{k}\right) \cdot \left(5\vec{\imath} + 4\vec{\jmath} + 3\vec{k}\right)}{\sqrt{50}} \right|$$

$$= \frac{1}{\sqrt{50}} = \frac{\sqrt{50}}{50} = \frac{\sqrt{2}}{10}$$

Example 2. Find the distance between $P = (3,5,6)$ and the line $\vec{X}(t) = \underbrace{(2,1,0)}_{Q} + t\underbrace{(1,5,9)}_{\vec{v}}$



$$d|\vec{v}| = \text{area}(A) = |\vec{QP} \times \vec{v}|$$

$$d = \left|\vec{QP} \times \frac{\vec{v}}{|\vec{v}|}\right|$$

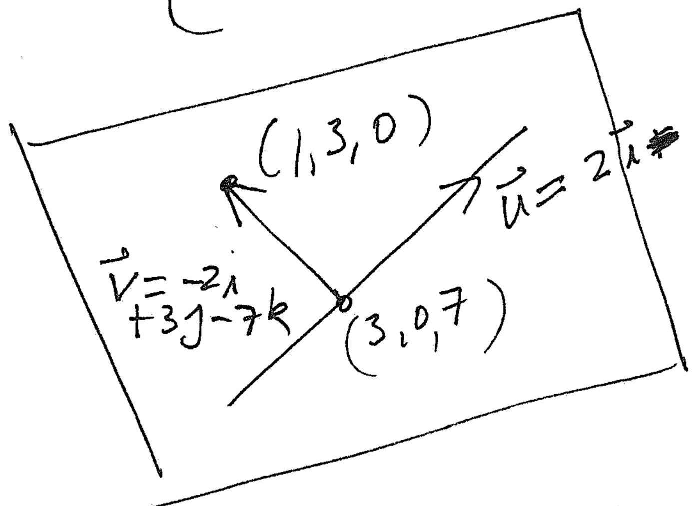
$$= \frac{|(1,4,6) \times (1,5,9)|}{\sqrt{107}}$$

$$= \frac{|(6,-3,1)|}{\sqrt{107}} = \sqrt{\frac{46}{107}}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 6 \\ 1 & 5 & 9 \end{vmatrix} = (6,-3,1)$$

Example 3 Find the equation of the plane containing $P=(1,3,0)$ and the line

$$\begin{cases} x = 3+2t \\ y = -4t \\ z = 7-t \end{cases}$$



$$\vec{X}(s,t) = \overrightarrow{(1,3,0)} + s(2\vec{i} - 4\vec{j} - \vec{k}) + t(-2\vec{i} + 3\vec{j} - 7\vec{k})$$

parametric form

implicit form: take $N = \vec{u} \times \vec{v}$

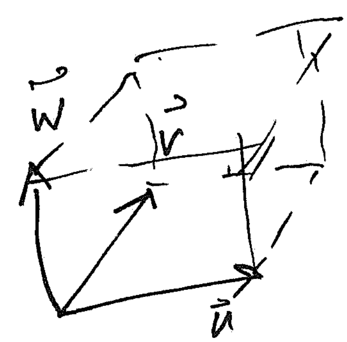
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & -1 \\ -2 & 3 & -7 \end{vmatrix} = 31\vec{i} + 16\vec{j} - 2\vec{k}$$

$$31(x-1) + 16(y-3) - 2z = 0$$

$$31x + 16y - 2z = 79$$

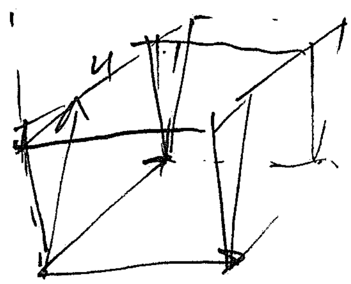
Scalar triple product, volume of a parallelpiped

3 vectors $\vec{u}, \vec{v}, \vec{w}$ span a "parallelpiped"



If you can visualize it, "slice and rearrange" this original papallelpiped to form a new parallelpiped "with base $\vec{u}, \vec{v}$" and a vector in the direction of $\vec{u} \times \vec{v}$

(formed by projecting the vector onto the $\frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|}$ direction)



The volume of the two parallelpipeds are equal to

$$V = \text{area base} \times \text{height}$$

$$= \| \vec{u} \times \vec{v} \| \, \| \text{Proj}_{\vec{u} \times \vec{v}} w \|$$

$$= \cancel{\| \vec{u} \times \vec{v} \|} \, \frac{\| \vec{w} \cdot \vec{u} \times \vec{v} \|}{\cancel{\| \vec{u} \times \vec{v} \|}}$$

$$= \| \vec{w} \cdot \vec{u} \times \vec{v} \|$$

The quantity $\vec{u} \times \vec{v} \cdot \vec{w}$

is called the "scalar triple product"

and can be written as a 3×3

determinant:

$$\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$$
$$\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$
$$w = w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k}$$

$$\vec{u} \times \vec{v} \cdot \vec{w} = \begin{Vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{Vmatrix}$$

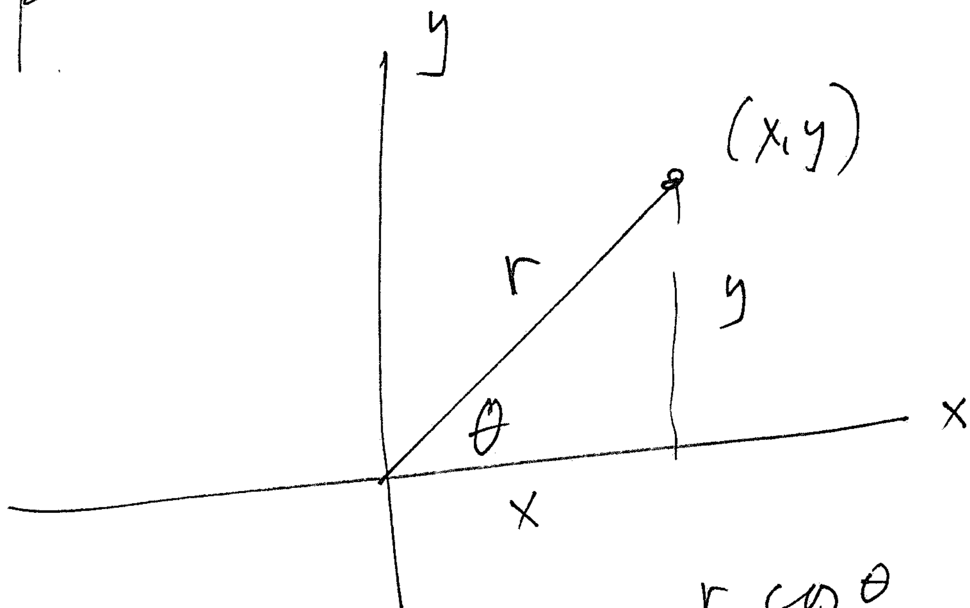
$$= w_1 \begin{Vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{Vmatrix} - w_2 \begin{Vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{Vmatrix} + w_3 \begin{Vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{Vmatrix}$$

$$\vec{w} \cdot \vec{u} \times \vec{v} = \vec{u} \cdot \vec{v} \times \vec{w} = \vec{v} \cdot (\vec{w} \times \vec{u})$$

invariant under "clockwise circular shift"

1.4 Cylindrical & spherical coordinates

polar coordinates in $\mathbb{R}^2$

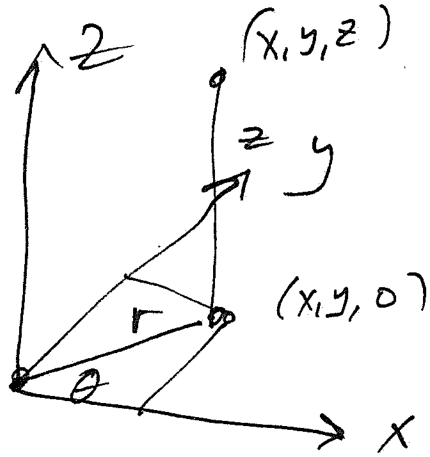


$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r = \sqrt{x^2+y^2}$$

cylindrical coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

polar coordinates for the projection of $\vec{x} = (x, y, z)$ onto the x, y plane

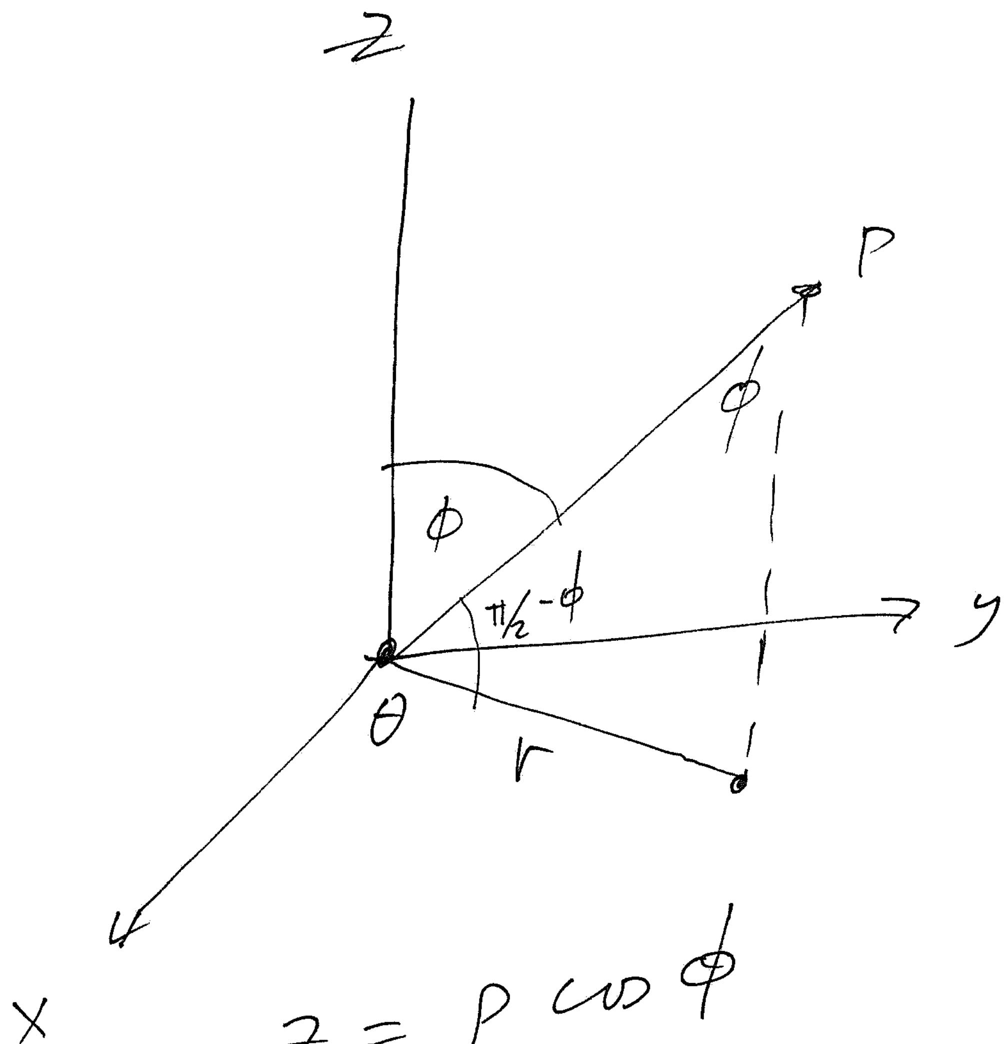
spherical coordinates

given a point P in R^3

we specify i) ρ = distance from P to the origin

ii) the angle ϕ between the positive z axis and the position vector of P

iii the polar angle θ of the projection of P onto the xy plane



$$z = \rho \cos \phi$$

$$r = \rho \sin \phi$$

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$