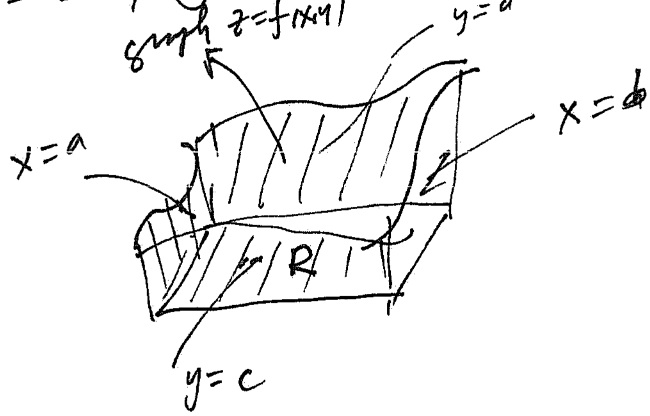
5.1 Introduction to the double integral

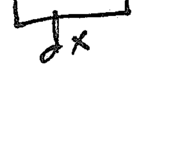
Consider a rectangle $R = [a,b] \times [c,d]$ in the x,y plane with sides parallel to the coordinate axes. Given a non-negative continuous function $f(x,y)$ on R, consider the region V in R^3 bounded above by the graph $z = f(x,y)$, below by R and on the sides by the planes $x=a$, $x=b$, $y=c$, $y=d$



We will in 5.2 discuss the method of Riemann sums to define the volume (V) by an approximation process of adding up small rectangular parallelepipeds.

The double integral ~~are~~ obtained is denoted $\iint_R f(x,y)\,dA$ or $\iint_R f(x,y)\,dx\,dy$

where we think of $dA = dx\,dy$ as the "infinitesimal ~~volu~~ area" of a rectangle with sides dx, dy



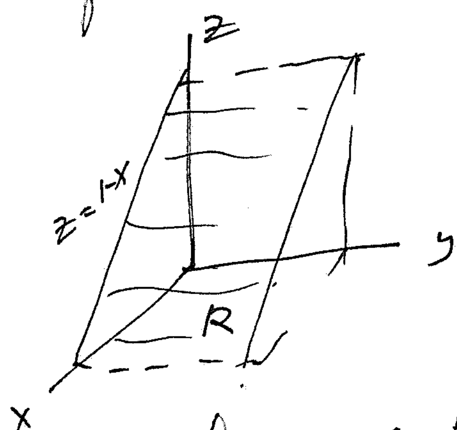
<u>Ex 1</u> a. If $f \equiv k$ constant > 0

then we must have $\iint_R f \, dA = k$ area R

$= k(b-a)(d-c)$

1b. If $f(x,y) = 1-x$, $R = [0,1] \times [0,1]$

then



$\iint_R f(x,y)\,dA =$

$\frac{1}{2} \cdot 1$ ($\frac{1}{2}$ of the volume of the rectangular box)

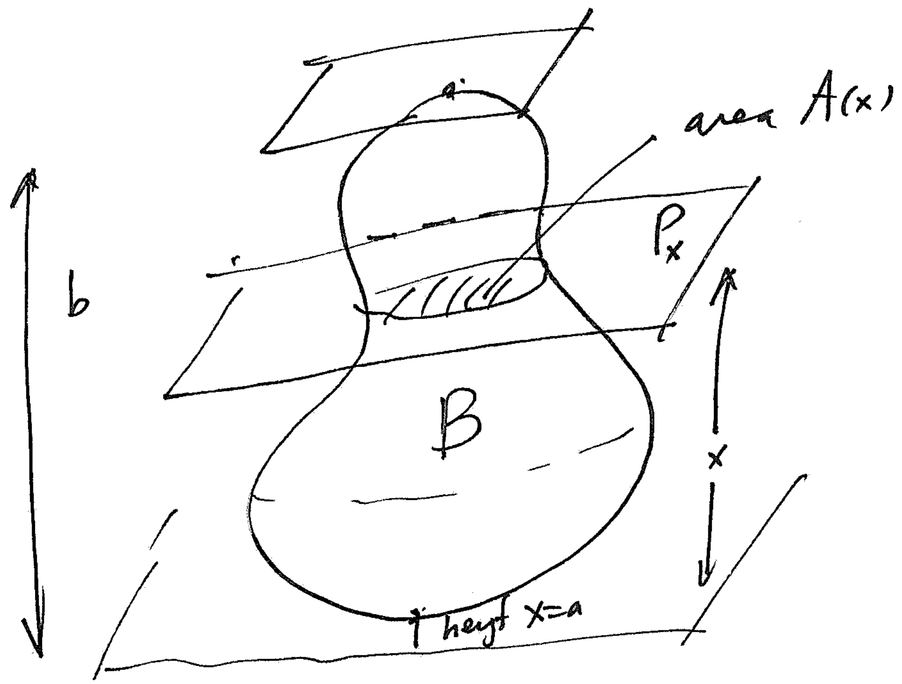
<u>Ex 2</u> Let $z = f(x,y) = x^2+y^2$
$R = [-1,1] \times [0,1]$

Then $\iint_R f(x,y)\,dA = \iint_R (x^2+y^2)\,dx\,dy$

represents the volume under the paraboloid $z = x^2 + y^2$ and above R.

There is a useful heuristic principle (which can be made rigorous by what is called Fubini's theorem) called <u>Cavalieri's principle</u>.

Consider a solid body B with the property that when cut by a plane P_x at distance x from a reference plane, it has cross-sectional area $A(x)$



Then Cavalieri's principle asserts

$$\text{volume}(B) = \int_a^b A(x)\,dx$$

Intuitively we partition $[a,b]$ into N equal (small) pieces $a = x_0 < x_1 < \dots < x_N = b$

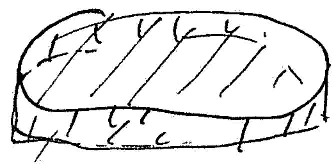
of size $\Delta x = x_i - x_{i-1}$ $\quad i = 1, \dots, N$

Then the Riemann sum approximating $\int_a^b A(x)\,dx$ is

$$\sum_{i=1}^{N} A(c_i)\,(x_i - x_{i-1}) = \sum_{i=1}^{N} A(c_i)\Delta x$$

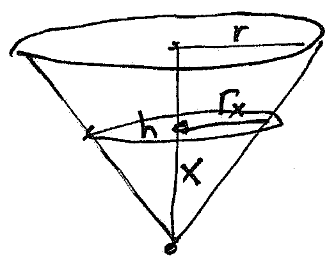
c_i a point in $[x_{i-1}, x_i]$

But $A(c_i)\,\Delta x$ is the infinitesimal volume of a "slab" with cross sectional area $A(c_i)$ and thickness Δx



and adding them up gives an approximation to the volume.

Ex (regular cone) C
ht h, base radius r



by similar triangles

$$\frac{x}{h} = \frac{r_x}{r}$$

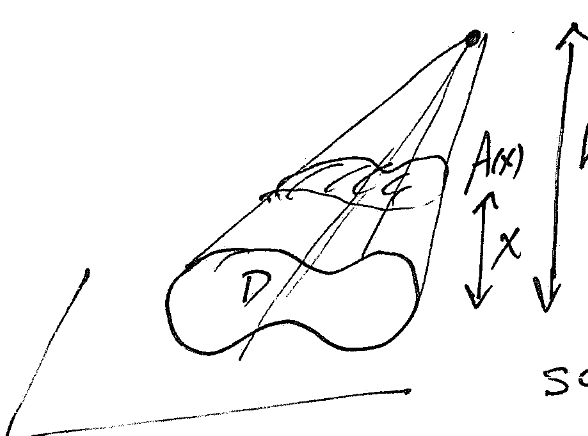
so $r_x = \frac{r\,x}{h}$

so $A(x) = \pi r_x^2 = \frac{\pi r^2}{h^2} x^2$

Hence $V(C) = \int_0^h \frac{\pi r^2}{h^2} x^2\,dx = \frac{1}{3}\frac{\pi r^2}{h^2} h = \frac{1}{3}\pi r^2 h$

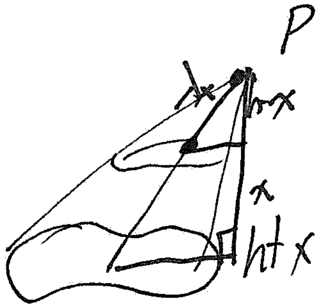
Ex (More sophisticated)

Suppose we choose a simple region D in the (x,y) plane of area A and an "apex point" P is at height h above the plane. We form the irregular pyramid with apex P formed by taking the cone over D. We need to compute $A(x)$



In order to do so we need the scaling factor λ_x for the section at height x

above the plane



again by similar triangles

$$\frac{h-x}{h} = \frac{\lambda_x}{1} \quad \text{so}$$

$$\lambda_x = \frac{h-x}{h} \quad \text{and so (area transform like } \lambda_x^2 \text{)}$$

$$A(x) = \lambda_x^2 A = \left(\frac{h-x}{h}\right)^2 A$$

Hence $\displaystyle Vol = \int_0^h \left(\frac{h-x}{h}\right)^2 A\, dx = \frac{A}{h^2}\int_0^h (h-x)^2\, dx$

$$= \frac{A}{h^2} \cdot -\frac{(h-x)^3}{3}\Big|_0^h = \frac{1}{3} A h$$

Reduction to Iterated Integrals

We now use Cavalieri's principle to evaluate double integrals on a rectangle $R = [a,b] \times [c,d]$

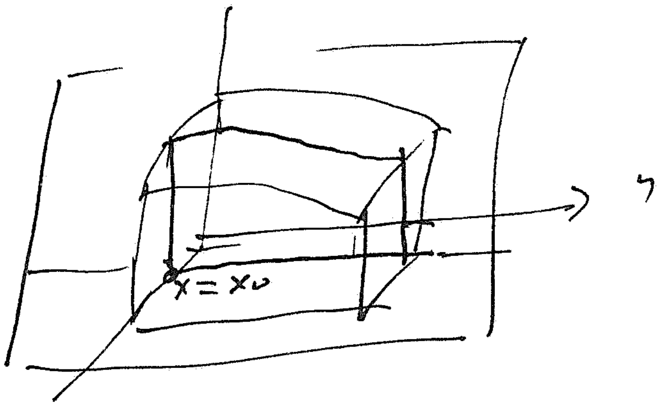
We can form cross-sectional area in two ways!

cut by the plane $x = x_0$, i.e. fix $x = x_0$ and obtain the function

$$y \mapsto f(x_0, y) \quad c \le y \le d$$

This has cross sectional area

$$\int_c^d f(x_0, y)\,dy$$



Then $V = \int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$

iterated integral with integration w.r.t. y first, then x

Similarly we can fix y_0 first and cut by the plane $y = y_0$ and obtain $V = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$

Hence $\iint_R f(x,y) = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$

$$= \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

<u>Ex</u> Evaluate $\iint_R (x^2+y^2)\,dx\,dy$

with $R = [-1,1] \times [0,1]$

<u>Solution</u> $\iint_R (x^2+y^2)\,dx\,dy = \int_0^1 \left[\int_{-1}^1 (x^2+y^2)\,dx \right] dy$

$$= \int_0^1 \left[\left(\frac{x^3}{3} + y^2 x \right) \Big|_{-1}^{1} \right] dy = \int_0^1 \left(\frac{2}{3} + 2y^2 \right) dy$$

$$= \left(\tfrac{2}{3} y + \tfrac{2}{3} y^3 \right) \Big|_0^1 = 4/3$$

__Example__ Evaluate $\iint_S \cos x \sin y \, dx\, dy$

where S is the square $[0, \pi/2] \times [0, \pi/2]$

__Solution__ $\iint_S \cos x \sin y \, dx\, dy =$

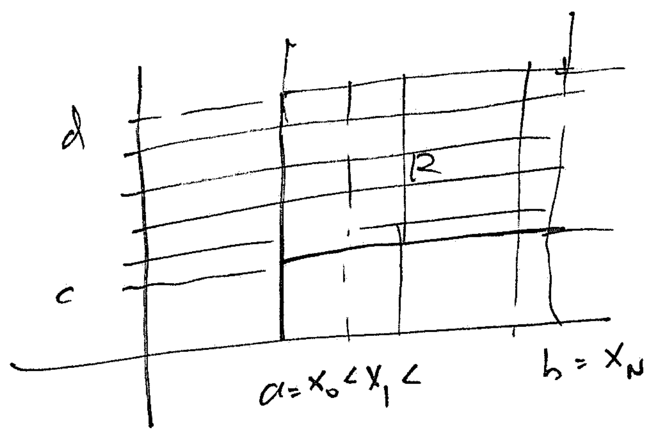
$$\int_0^{\pi/2} \left[\int_0^{\pi/2} \cos x \sin y \, dx \right] dy$$

$$= \int_0^{\pi/2} \sin y \left(\underbrace{\int_0^{\pi/2} \cos x \, dx}_{=1} \right) dy$$

$$= \int_0^{\pi/2} \sin y \, dy = 1$$

5.2 The double integral over a rectangle

We start with a "regular partition" of $R = [a,b] \times [c,d]$,



that $x_{i+1} - x_i = \frac{b-a}{N}$, $y_{j+1} - y_j = \frac{d-c}{N}$

We will always assume $f(x,y)$ is bounded on R i.e. $-M \leq f(x,y) \leq M$

Moreover we will assume

either f is continuous ~~and~~ or perhaps (sometimes) the singularities of f are "small" in measure.

In the rectangle $R_{jk} = [x_j, x_{j+1}] \times [y_k, y_{k+1}]$ we let c_{jk} be any point where f is continuous and we form the sum

$$S_N = \sum_{j,k=0}^{N-1} f(c_{jk})\,\Delta x\,\Delta y = \sum_{j,k=0}^{N-1} f(c_{jk})\,\Delta A$$

where $\Delta x = x_{j+1} - x_j = \frac{b-a}{N}$, $\Delta y = y_{k+1} - y_k = \frac{d-c}{N}$

and $\Delta A = \Delta x\,\Delta y$

The sum S_N has N^2 terms

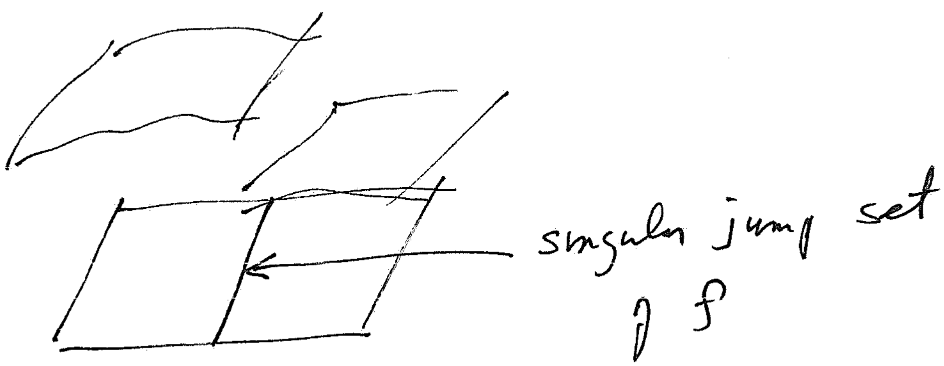
<u>Def'n</u> If the sequence $\{S_N\}$ tends to a limit as $N \to \infty$ (independent of choice of c_{jk}) we say f is integrable over R and we write $\iint_R f(x,y)\,dA := \int_R f(x,y)\,dx\,dy$

<u>Theorem 1</u> A continuous function on R is integrable

A more sophisticated argument yields a more general statement

<u>Theorem 2</u> Let $f : R \to \mathbb{R}$ be bounded on a rectangle R with singularities lying on a finite union of continuous graphs. Then f is integrable over R

<u>Ex</u> (f has a "jump discontinuity" on a curve)



Properties of the Integral

i) $$\iint_R (f+g)\, dA = \iint_R f\, dA + \iint_R g\, dA$$

(f, g integrable on R)

ii) $$\iint_R c f\, dA = c \iint_R f\, dA$$

iii) If $f \geq g$ then

$$\iint_R f\, dA \geq \iint_R g\, dA$$

iv) If R_i, $i = 1, \dots, m$ are pairwise disjoint and $R = \bigcup_{i=1}^{m} R_i$, then

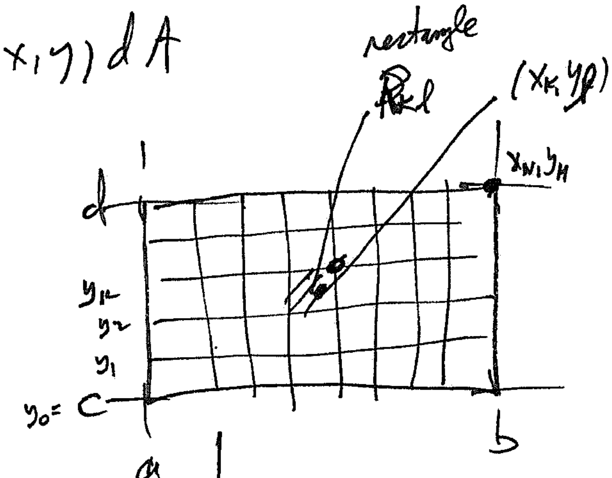
$$\iint_R f\, dA = \sum_{i=1}^{m} \iint_{R_i} f\, dA$$

Fubini's theorem

Let f be continuous on $R = [a,b] \times [c,d]$

Then $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$

$$= \iint_R f(x,y)\,dA$$



idea: N rectangles in x direction
M " " y "

$$R = \sum_{\substack{1 \le k \le N \\ 1 \le l \le M}} f(x_k, y_l)\,\Delta x\,\Delta y$$

(x_k, y_l) = upper right corner in rectangle in the k^{th} rectangle from left and l^{th} row from above

$$\begin{aligned} = \; & f(x_1,y_1)\,\Delta x\,\Delta y + \dots + f(x_N,y_1)\,\Delta x\,\Delta y \\ + \; & f(x_1,y_2)\,\Delta x\,\Delta y + \dots + f(x_N,y_2)\,\Delta x\,\Delta y \\ & \vdots \\ + \; & f(x_1,y_M)\,\Delta x\,\Delta y + \dots + f(x_N,y_M)\,\Delta x\,\Delta y \end{aligned}$$

*

$x_k = a + k\,\Delta x$

$y_l = c + l\,\Delta y$

row # k of * $= \Delta y \{ f(x_1,y_k)\,\Delta x + f(x_2,y_k)\,\Delta x + \dots + f(x_N,y_k)\,\Delta x \}$

Δy (Riemann sum for $\int_a^b f(x,y_k)\,dx$)

So if $I(y) = \int_a^b f(x,y)\,dx$

row # k $\nearrow$ = $\Delta y \times I(y_k)$

so $R \simeq I(y_1)\Delta y + I(y_2)\Delta y + \cdots + I(y_n)\Delta y$

which is a Riemann sum for $\int_c^d I(y)\,dy$

Hence $R \simeq \int_c^d I(y)\,dy$

Taking limits as the partition sizes $\to 0$

$$\Rightarrow \iint_R f(x,y)\,dA = \int_c^d I(y)\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right]$$

Similarly using the columns

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Example 1: Compute $\iint_R (x^2+y)\,dA$

$R = [0,1] \times [0,1]$

By Fubini,

$$\iint_R (x^2+y)\,dA = \int_0^1 \left(\int_0^1 x^2+y\,dx \right) dy$$

$$= \int_0^1 \left(\frac{x^3}{3}+yx \right)\Big|_0^1 dy = \int_0^1 \left(\frac{1}{3}+y\right) dy$$

$$= \left(\tfrac{1}{3}y + \frac{y^2}{2} \right)\Big|_0^1 = \frac{1}{3}+\frac{1}{2} = \frac{5}{6}$$

Ex 2 in reverse order

$$\iint_R (x^2+y)\,dA = \int_0^1 \left(\int_0^1 (x^2+y)\,dy \right) dx$$

$$= \int_0^1 \left(x^2 y + \frac{y^2}{2} \right)\Big|_0^1 dx = \int_0^1 \left(x^2+\frac{1}{2}\right) dx$$

$$= \left(\frac{x^3}{3} + \frac{1}{2}x \right)\Big|_0^1 = \tfrac{1}{3}+\tfrac{1}{2} = \tfrac{5}{6}$$

Ex $\iint_R x e^{xy}\, dA$, $R = [-1,2]\times[0,1]$

$$= \int_{-1}^{2}\left(\int_0^1 x e^{xy}\, dy\right) dx$$

$$= \int_{-1}^{2} e^{xy}\Big|_0^1\, dx = \int_{-1}^{2} (e^x - 1)\, dx$$

$$= (e^x - x)\Big|_{-1}^{2} = (e^2 - 2) - (e^{-1} + 1)$$

$$= e^2 - e^{-1} - 3$$

The other ~~way~~ order is harder:

$$\int_0^1\left(\int_{-1}^{2} x e^{xy}\, dx\right) dy = \int_0^1 \left\{\left(\frac{2}{y} - \frac{1}{y^2}\right)e^{2y} + \left(\frac{1}{y} + \frac{1}{y^2}\right)e^{-y}\right\} dy$$

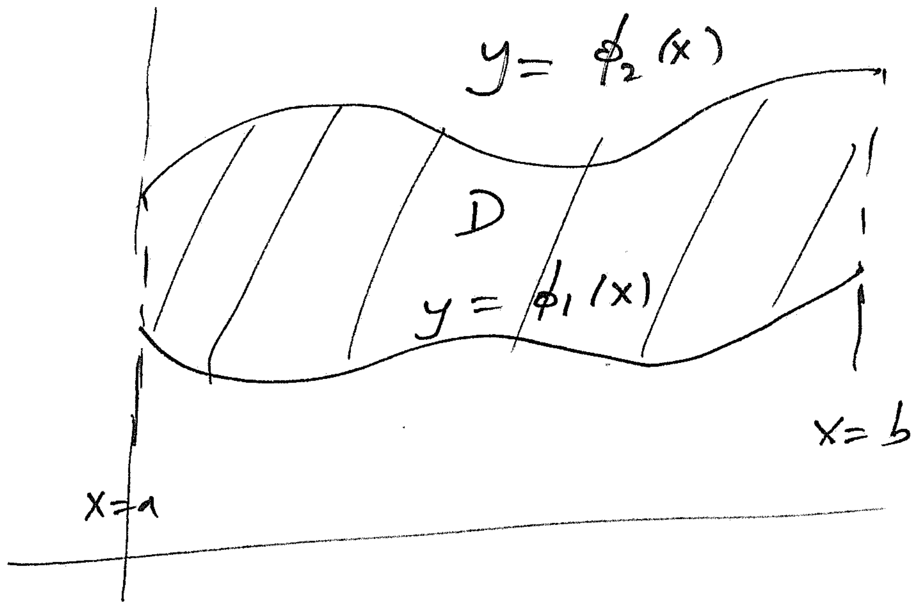
$$\int_{-1}^{2} x e^{xy}\, dx = x\,\frac{e^{xy}}{y}\Big|_{-1}^{2} - \int_{-1}^{2} \frac{e^{xy}}{y}\, dx =$$

$$\frac{2e^{2y} + e^{-y}}{y} - \frac{1}{y^2} e^{xy}\Big|_{-1}^{2}$$

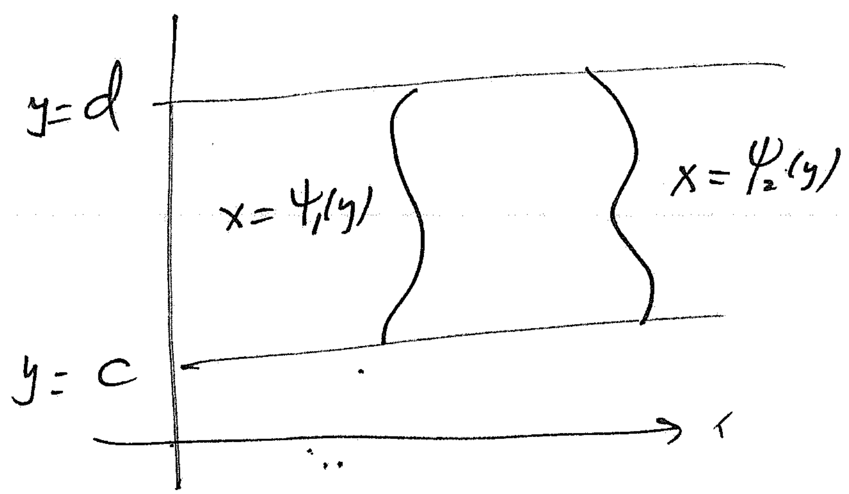
$$= \frac{2e^{2y} + e^{-y}}{y} - \frac{1}{y^2}\left(e^{2y} - e^{-y}\right)$$

5.3 & 5.4 The double integral over elementary regions, change of order

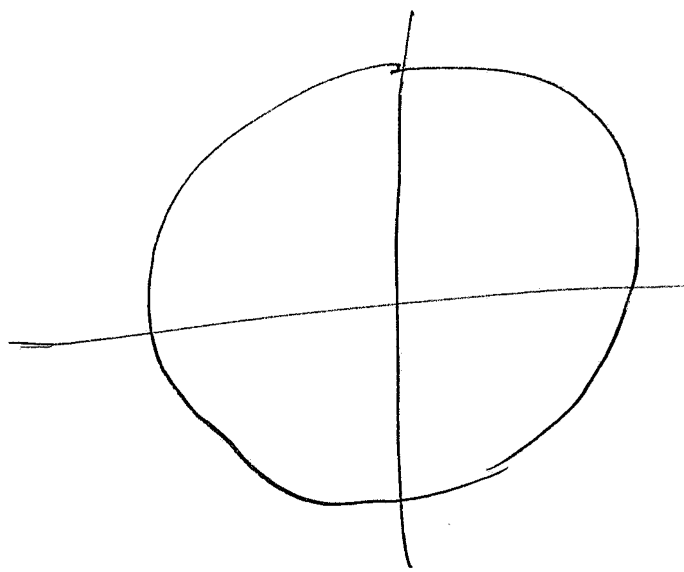
y-simple region D



x-simple region D



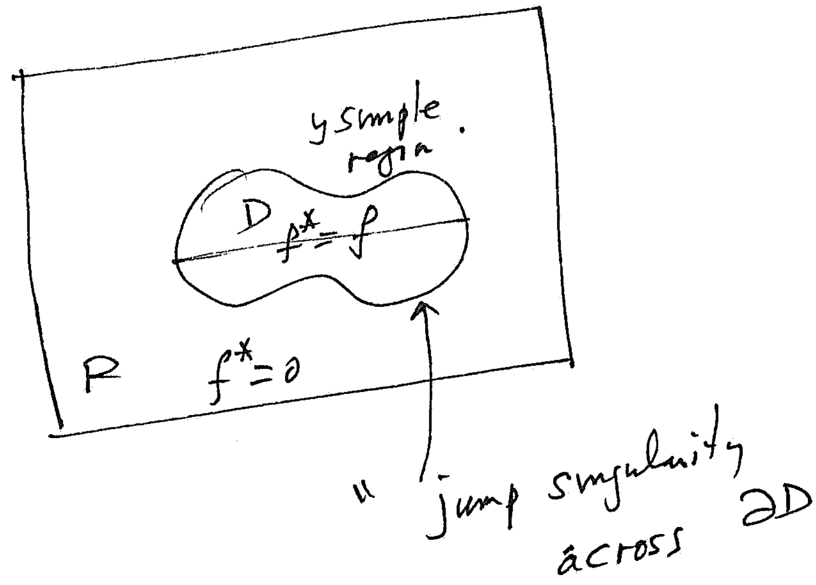
a simple region D is one that is both x simple and y simple



the unit disk is both x simple and y simple

Define $f^*(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \notin D \\ & (x,y) \in R \end{cases}$

where R is a rectangle which contains D in its interior



Then

$$\iint_D f(x,y)\,dA = \iint_R f^*(x,y)\,dA$$

$$= \int_a^b \int_c^d f^*(x,y)\,dy\,dx = \int_c^d \int_a^b f^*(x,y)\,dx\,dy$$

<u>If D is y simple</u>

$$\int_c^d f^*(x,y)\,dy = \int_{\phi_1(x)}^{\phi_2(x)} f^*(x,y)\,dy = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy$$

So

<u>Thm</u> i) If D is y-simple, f continuous on D

$$\iint_D f(x,y)\,dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx$$

ii) Similarly if D is x-simple

$$\iint_D f(x,y)\,dA = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy$$

<u>Remark</u> If $f \equiv 1$ Then

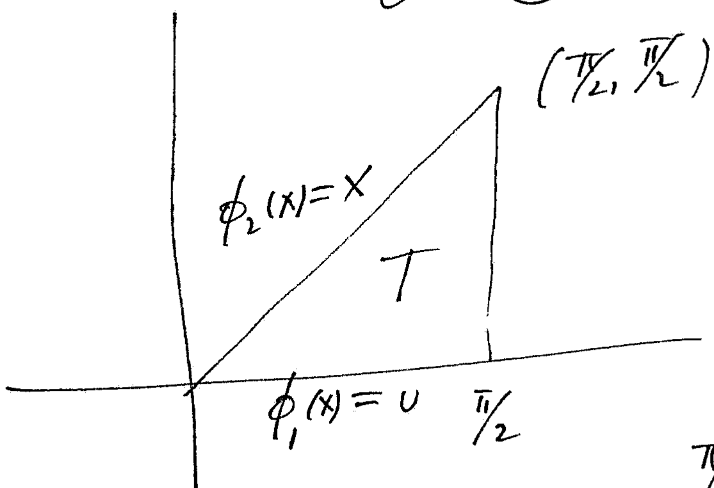
$$\iint_D f\,dA = A(D) = \int_a^b (\phi_2(x) - \phi_1(x))\,dx$$

as in Calculus I

Ex 1 Evaluate $\iint_T (x^3 y + \cos x)\, dA$

where T is the triangle:

$$0 \le x \le \pi/2$$
$$0 \le y \le x$$



$$\iint_T (x^3 y + \cos x)\, dA = \int_0^{\pi/2} \int_0^{x} (x^3 y + \cos x)\, dy\, dx$$

$$= \int_0^{\pi/2} \left\{ \frac{x^3 y^2}{2} + \cos x \cdot y \right\}_0^{x} dx$$

$$= \int_0^{\pi/2} \left(\frac{x^5}{2} + x\cos x\right) dx = \frac{x^6}{12}\Big|_0^{\pi/2} + \int_0^{\pi} x \cos x \, dx$$

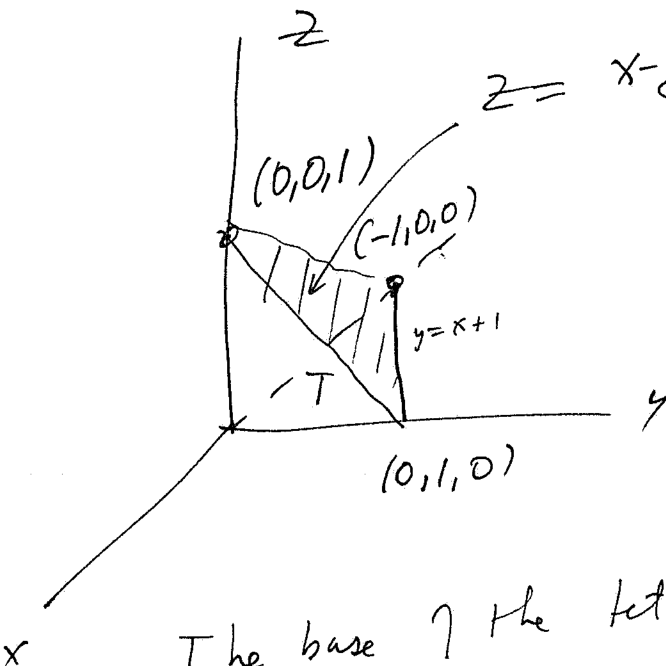
$$= \frac{\pi^6}{768} + x\sin x\Big|_0^{\pi/2} - \int_0^{\pi} \sin x\,dx$$

$$= \frac{\pi^6}{768} + \frac{\pi}{2} + \cos x\Big|_0^{\pi/2} = \frac{\pi^6}{768} + \pi/2 - 1$$

Example 2

Find the volume of the tetrahedron bounded by the planes $y=0$, $z=0$, $x=0$ and $y-x+z=1$

$z = x-y+1$



when $z=0$
$y-x=1$, $y=x+1$

when $x=0$
$y+z=1$

when $y=0$ $z=x+1$

The base of the tetrahedron is the triangle T in the (x,y) plane: T is y simple region

$$-1 \le x \le 0$$
$$0 \le y \le x+1$$

$$\text{Volume} = \iint_T (x-y+1)\,dA = \int_{-1}^{0}\int_{0}^{x+1} (x-y+1)\,dy\,dx$$

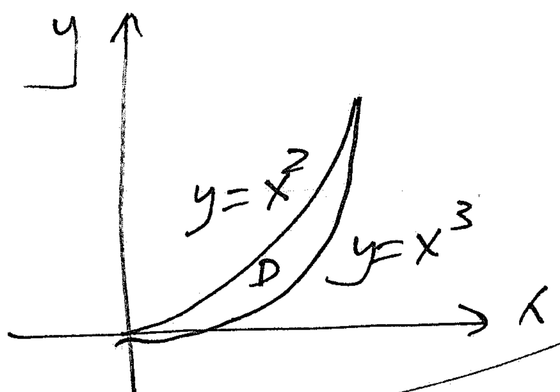
$$= \int_{-1}^{0} \left\{ \left[(x+1)y - \frac{y^2}{2} \right]_0^{x+1} \right\} dx = \frac{1}{2}\int_{-1}^{0} (x+1)^2\, dx$$

$$= \frac{1}{6}(x+1)^3 \Big|_{-1}^{0} = \frac{1}{6}(1-0) = \frac{1}{6}$$

Example 3 (y-simple region)

Evaluate $\int_0^1 \int_{x^3}^{x^2} xy \, dy \, dx$

and sketch the region D

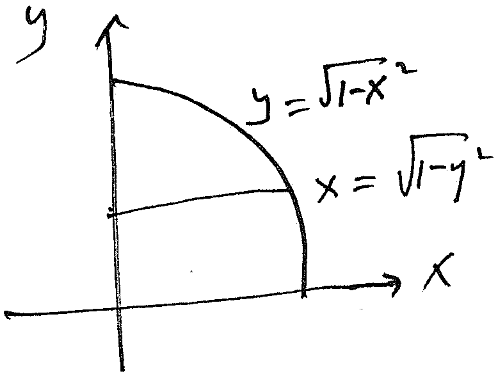


$$= \int_0^1 \left(\frac{xy^2}{2} \Big|_{y=x^3}^{y=x^2} \right) dx = \int_0^1 \left(\frac{x^5}{2} - \frac{x^7}{2} \right) dx$$

$$= \frac{1}{12} - \frac{1}{16} = \frac{1}{48}$$

Example 4 Reverse the order of integration (change to x simple region)

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy\, dx$$



$$= \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\, dx\, dy$$

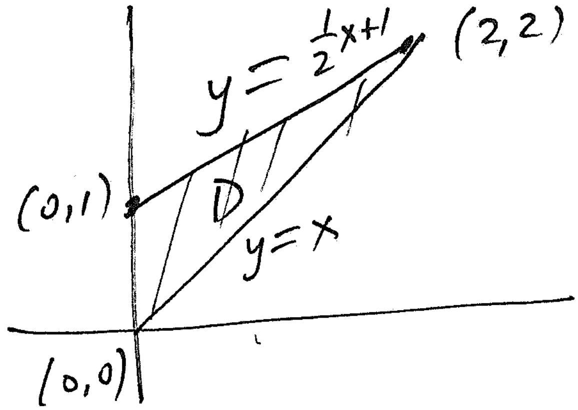
$$= \int_0^1 \left[\sqrt{1-y^2} \cdot \sqrt{1-y^2}\right] dy$$

$$= \int_0^1 (1-y^2)\, dy = 1 - \frac{1}{3} = \frac{2}{3}$$

Example 5 Calculate the integral

$f(x,y) = (x+y)^2$

over the region D:



$$\iint_D f(x,y)\,dx\,dy = \int_0^2 \int_x^{\frac{1}{2}x+1} (x+y)^2\,dy\,dx$$

$$= \int_0^2 \left[\frac{1}{3}(x+y)^3 \Big|_{y=x}^{y=\frac{1}{2}x+1} \right] dx$$

$$= \frac{1}{3}\int_0^2 \left[\left(\frac{3}{2}x+1\right)^3 - (2x)^3 \right] dx$$

$$= \frac{1}{3}\left[\frac{1}{6}\left(\frac{3}{2}x+1\right)^4 - 2x^4 \right]_0^2$$

$$= \frac{1}{3}\left[\frac{1}{6}(4^4-1) - 2\cdot 16 \right]$$

$$= \frac{1}{3}\left(\frac{255}{6} - 32 \right) = \frac{1}{3}\left(\frac{85-64}{2} \right)$$

$$= \frac{7}{2}$$

5.4 Changing the order of integration

If D is simple, that is, both x simple & y simple

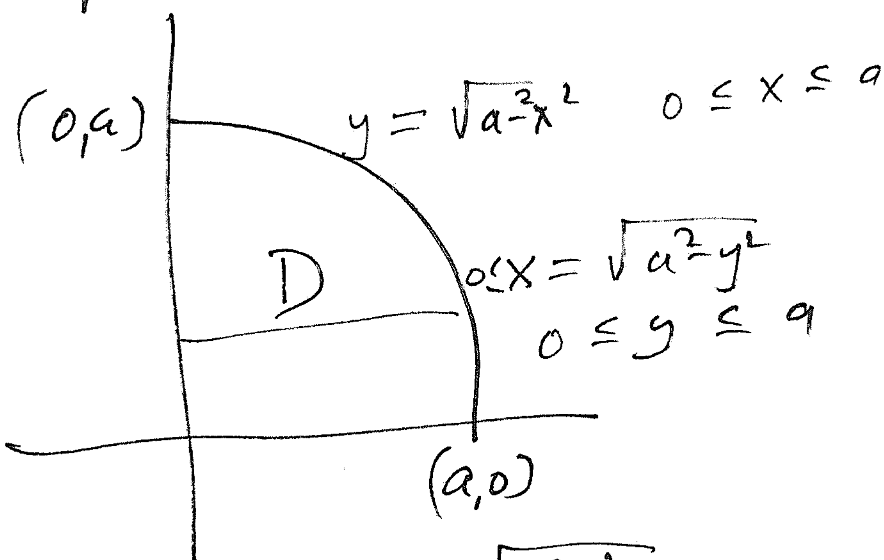
$$\iint_D f\, dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\, dy\, dx$$

$$= \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx\, dy$$

Example 1 Evaluate

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-y^2}\, dy\, dx$$

by changing the order of integration



$$= \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2}\, dx \right] dy$$

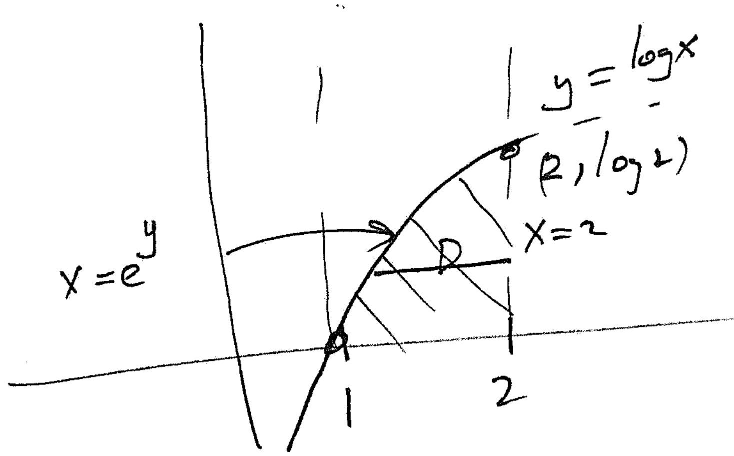
$$= \int_0^a \left[x\sqrt{a^2-y^2} \Big|_0^{\sqrt{a^2-y^2}} \right] dy$$

$$= \int_0^a (a^2-y^2)\, dy = \left(a^2 y - \frac{y^3}{3} \right)\Big|_0^a$$

$$= \frac{2}{3} a^3$$

Example 2

evalute $\int_1^2 \int_0^{\log x} (x-1)\sqrt{1+e^{2y}}\, dy\, dx$



$$= \int_0^{\log 2} \sqrt{1+e^{2y}} \left[\int_{e^y}^{2} (x-1)\, dx \right] dy$$

$$= \int_0^{\log 2} \sqrt{1+e^{2y}} \left(\frac{x^2}{2} - x \right) \Big|_{e^y}^{2} dy$$

$$= \int_0^{\log 2} \sqrt{1+e^{2y}} \left(e^y - \frac{e^{2y}}{2} \right) dy$$

$$= \int_0^{\log 2} e^y \sqrt{1+e^{2y}}\,dy \;-\; \frac{1}{2}\int_0^{\log 2} \sqrt{1+e^{2y}}\,e^{2y}\,dy$$

$v = e^y$ $\qquad u = e^{2y}$

$$= \int_1^2 \sqrt{1+v^2}\,dv \;-\; \frac{1}{4}\int_1^4 (1+u)^{1/2}\,du$$

$$\frac{1}{4}\cdot\frac{2}{3}(1+u)^{3/2}\Big|_1^4$$

$$= \frac{1}{6}\left(5^{3/2} - 2^{3/2}\right)$$

$\cancel{v = \tan\theta}$

$\cancel{dv = \sec^2\theta\, d\theta}$

$\cancel{= \int \sec^3\theta\, d\theta = \int}$

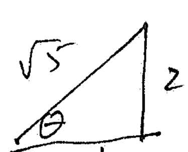
integrate by parts: $\sqrt{1+v^2}\cdot v\Big|_1^2 = 2\sqrt{5} - \sqrt{3}$

$$-\int_1^2 \frac{v^2}{\sqrt{1+v^2}}\,dv = \int_1^2 \frac{1}{\sqrt{1+v^2}}\,dv \quad v = \tan\theta$$

$$- \int_1^2 \sqrt{1+v^2}\,dv$$

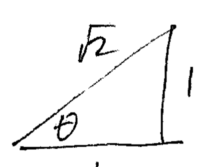
$$2\int_1^2 \sqrt{1+v^2}\,dv = 2\sqrt{5} - 3$$

$$+ \int_{\tan^{-1}1}^{\tan^{-1}2} \sec\theta\, d\theta = \log(\sec\theta + \tan\theta)\Big|_{\tan^{-1}1}^{\tan^{-1}2}$$



$\tan\theta = 2$

$\sec\theta = \sqrt{5}$



$\tan\theta = 1$

$\sec\theta = \sqrt{2}$

$$= \log\left(\frac{\sqrt{5}+2}{1+\sqrt{2}}\right)$$

36

$$= \frac{1}{2}\left\{ 2\sqrt{5} - 3 + \log\left(\frac{\sqrt{5}+2}{1+\sqrt{2}}\right) \right\} - \frac{1}{6}\left(5^{3/2} - 2^{3/2}\right)$$

5.5 The triple integral

We want to define the triple integral $\iiint_B f(x,y,z)\,dV$

over a box (rectangular parallelpiped)

$B = [a,b] \times [c,d] \times [p,q]$

by Riemann sums. We do not emphasize the interpretation for $f \geq 0$

as a volume because the region below the graph of f

is now in $\mathbb{R}^4$.

By analogy we partition the three sides of B into N equal parts and form the sum

$$S_N = \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} f(c_{ijk}) \, \Delta V$$

where c_{ijk} is a point in the ijk th partition box B_{ijk} with volume ΔV, and $f(x,y,z)$ is a continuous function.

Def'n If $\lim_{N\to\infty} S_N$ exists,

independent of any choice of c_{ijk}

we say f is integrable, and

call $\iiint_B f\,dV = \iiint_B f(x,y,z)\,dx\,dy\,dz$

the triple integral of f

As before it turns out that

if f is bounded with jump

singularities confined to graphs

of continuous functions, e.g

$x = \alpha(y,z)$, $y = \beta(x,z)$, $z = \gamma(x,y)$,

then $\iiint f\,dV$ exists. The

other basic properties remain the

same. Moreover,

$$\iiint_B f(x,y,z)\,dV = \int_p^q \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

$$= \int_p^q \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz$$

$$= \int_a^b \int_p^q \int_c^d f(x,y,z)\,dy\,dz\,dx$$

and so on (6 possible orders)

Example 1 Let $B = [0,1]\times[-1/2,0]\times[0,1/3]$

Evaluate $\displaystyle\iiint_B (x+2y+3z)^2\,dx\,dy\,dz$

$$= \int_0^{1/3}\int_{-1/2}^{0}\int_0^1 (x+2y+3z)^2\,dx\,dy\,dz$$

$$= \int_0^{1/3}\int_{-1/2}^{0}\left[\frac{(x+2y+3z)^3}{3}\Bigg|_{x=0}^{x=1}\right] dy\,dz$$

$$= \frac{1}{3}\int_0^{1/3}\int_{-1/2}^{0}\left[(1+2y+3z)^3-(2y+3z)^3\right] dy\,dz$$

$$\cancel{\tfrac{1}{2}[(1+4y+6z)^2}$$

$$= \frac{1}{3\cdot4\cdot2}\int_0^{1/3}\left[\left((1+2y+3z)^4-(2y+3z)^4\right)\Bigg|_{y=-1/2}^{y=0}\right] dz$$

$$= \frac{1}{24}\int_0^{1/3}\left[(1+3z)^4-(3z)^4-(3z)^4+(-1+3z)^4\right] dz$$

$$= \frac{1}{24\cdot15}\left[(1+3z)^5+(3z-1)^5-2\cdot(3z)^5\right]\Bigg|_0^{1/3}$$

$$= \frac{1}{24\cdot15}\left(2^5-2\right) = 1/12$$

Example 2 $\iiint_B e^{x+y+z}\,dV$,

$B = [0,1]\times[0,1]\times[0,1]$

Integrating in standard order:

$$\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz = \int_0^1\int_0^1 e^{y+z}(e-1)\,dy\,dz$$

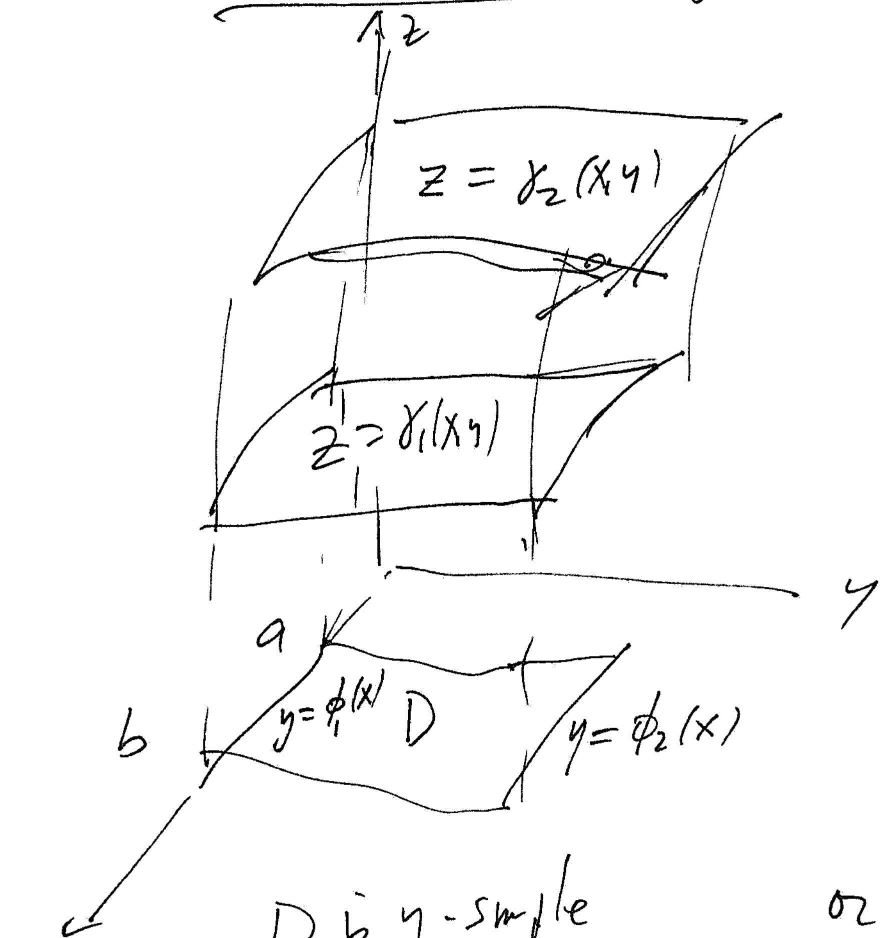
$$= (e-1)^2\int_0^1 e^z\,dz = (e-1)^3. \;//$$

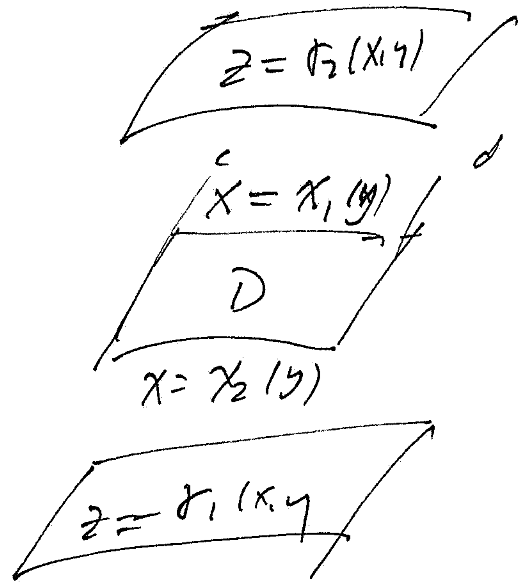
As before we define $\iiint_W f\,dV$

$= \iiint_B f^* \,dV$ where $W \subset\subset B$

and $f^* = \begin{cases} f & \text{in } W \\ 0 & B-W \end{cases}$

Elementary regions





D is y-simple
$a \le x \le b$

or perhaps

D is x simple
$c \le y \le d$

Ex

$$x^2 + y^2 + z^2 \le 1$$

$$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$$

$$\left.\begin{aligned} -\sqrt{1-x^2} &\le y \le \sqrt{1-x^2} \\ -1 &\le x \le 1 \end{aligned}\right\} \text{D is y-simple}$$

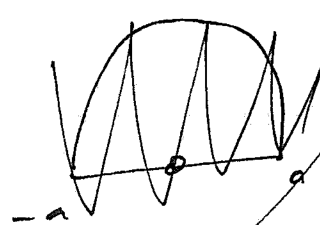
We find the volume of the unit ball $= \frac{4}{3}\pi$:

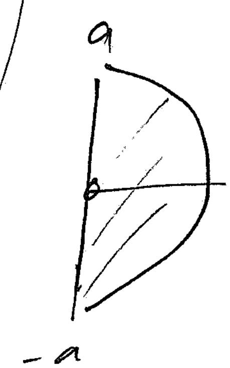
$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

$$= 2 \int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy \right] dx =$$

with x fixed, set $a = \sqrt{1-x^2}$

$$\int_{-a}^{a} \sqrt{a^2-y^2}\,dy = \frac{1}{2}\pi a^2 = \frac{\pi}{2}(1-x^2)$$

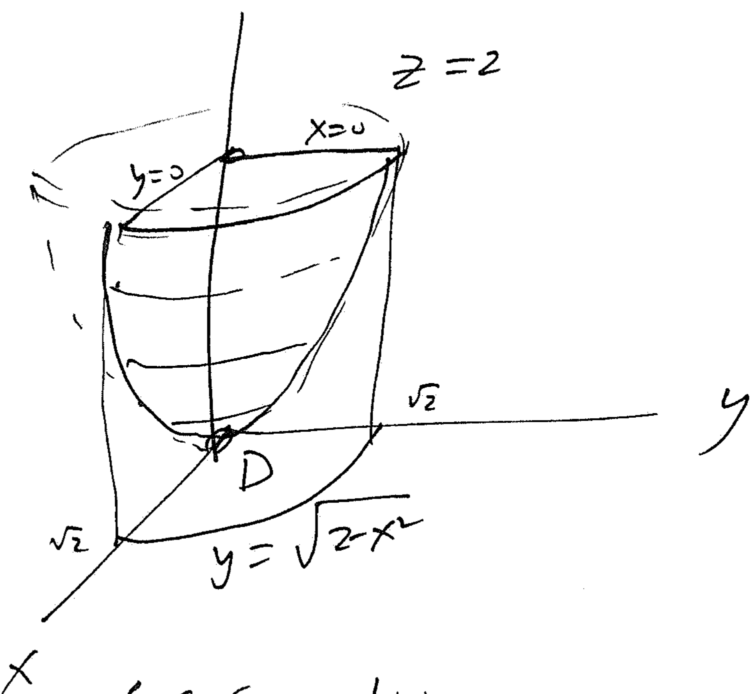




$$2 \cdot \frac{\pi}{2} \int_{-1}^{1} (1-x^2)\,dx = \pi \left(x - \frac{x^3}{3} \right) \Big|_{x=-1}^{x=1}$$

$$= \frac{4}{3}\pi \quad //$$

Ex 5 W = region bounded by the planes $x=0$, $y=0$, $z=2$ and the surface $z=x^2+y^2$ over the first quadrant $x, y \geq 0$



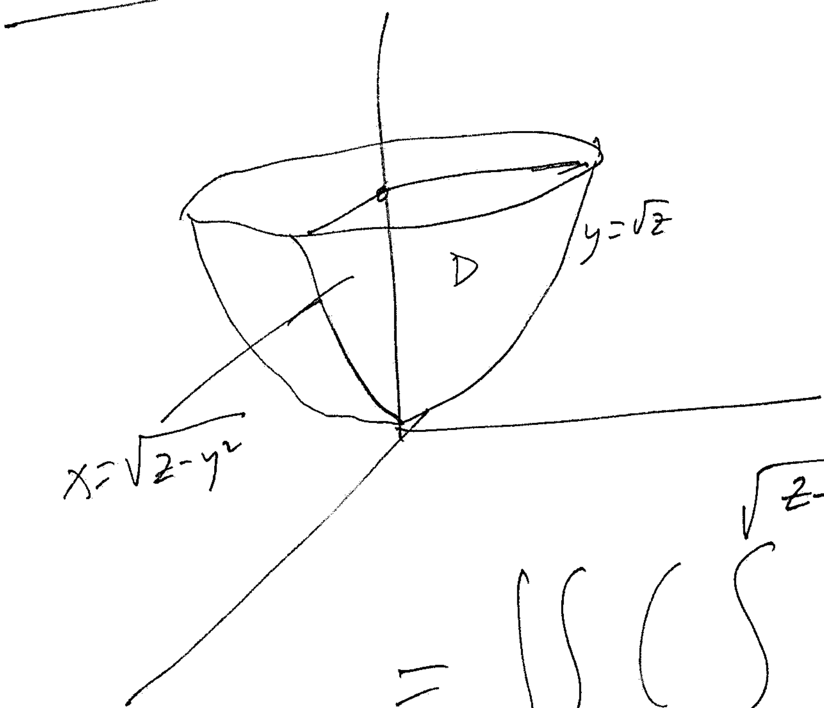
Compute $\iiint_W x\,dV =$

$$= \int_0^{\sqrt{2}} \left[\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x\,dz \right) dy \right] dx = \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} x(2-x^2-y^2)\,dy\,dx$$

$$= \int_0^{\sqrt{2}} x\left((2-x^2)^{3/2} - \frac{(2-x^2)^{3/2}}{3} \right) dx = \frac{2}{3}\int_0^{\sqrt{2}} x(2-x^2)^{3/2}\,dx$$

$$= \frac{1}{3}\,\frac{(2-x^2)^{5/2}}{5/2}\Big|_0^{\sqrt{2}} = \frac{2}{15}\cdot 2^{5/2} = \frac{8\sqrt{2}}{15}$$

method 2



$$\iiint_W x\,dV$$

$$= \iint_D \left(\int_0^{\sqrt{z-y^2}} x\, d\cancel{z}x \right) dy\, dz$$

$$= \int_0^2 \left[\int_0^{\sqrt{z}} \frac{z-y^2}{2}\, dy \right] dz$$

$$= \frac{1}{2} \int_0^2 \left[\left(zy - \frac{y^3}{3} \right) \Big|_{y=0}^{y=\sqrt{z}} \right] dz$$

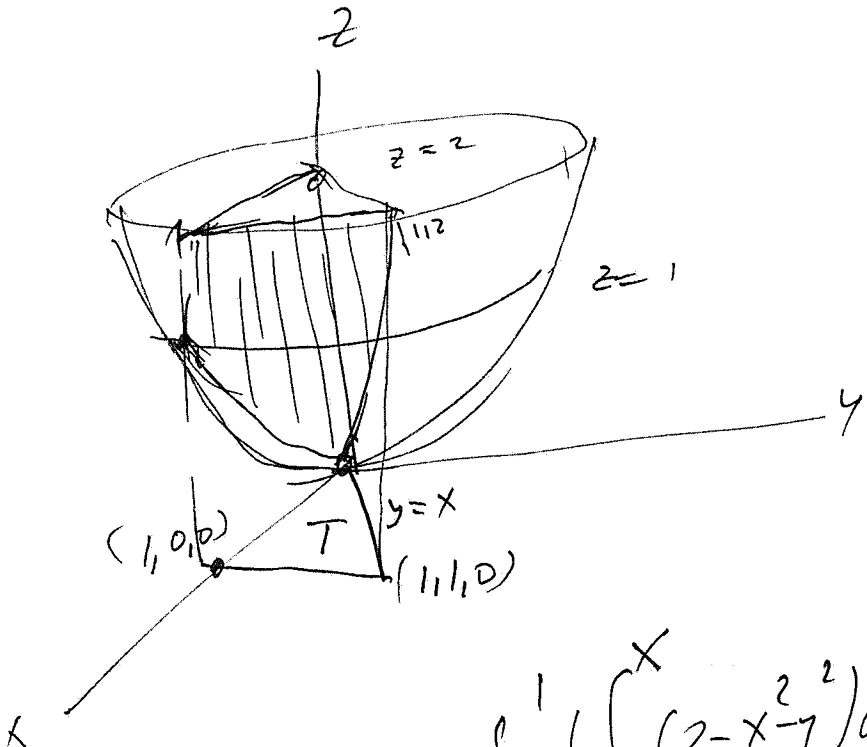
$$= \frac{1}{2} \int_0^2 \left(z^{3/2} - \frac{z^{3/2}}{3} \right) dz = \frac{1}{3} \frac{z^{5/2}}{5/2} \Big|_0^2 = \frac{2}{15} 2^{5/2}$$

$$= \frac{2\sqrt{32}}{15} = \frac{8\sqrt{2}}{15}$$

Ex Evaluate $\int_0^1 \int_0^x \int_{x^2+y^2}^2 dz\,dy\,dx$ (46)

and sketch the region W

$$x^2+y^2 \le z \le 2$$
$$0 \le y \le x$$
$$0 \le x \le 1$$



$y=0 \quad z = x^2$

$$= \int_0^1 \left(\int_0^x (2 - x^2 - y^2)\,dy \right) dx$$

$$= \int_0^1 \left(\left(2x - x^3 - \frac{x^3}{3}\right) \Big|_0^x \right) dx$$

$$= \int_0^1 \left(2x - \tfrac{4}{3}x^3\right) dx = 1 - \tfrac{1}{3} = \tfrac{2}{3}$$

Ex Rewrite the iterated integral

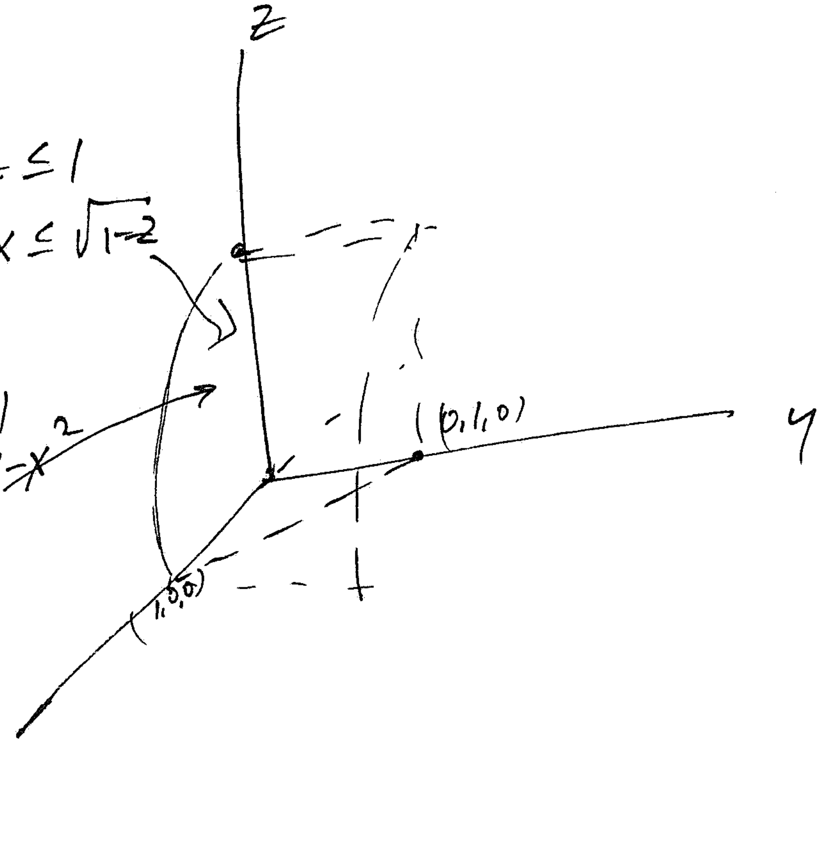
$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx$$

in the order $dy\,dx\,dz$

W: $0 \le x \le 1$
$0 \le z \le 1-x^2$
$0 \le y \le 1-x$

$0 \le z \le 1$
$0 \le x \le \sqrt{1-z}$

$0 \le x \le 1$
$0 \le z \le 1-x^2$



$$= \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f \, dy\, dx\, dz$$

$$= \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f \, dz\, dx\, dy$$