# The Change of variable formula for one variable calculus integrals

$$(*) \quad A(t) := \int_a^t f(\varphi(x))\, \varphi'(x)\, dx = \int_{\varphi(a)}^{\varphi(t)} f(x)\, dx =: B(t)$$

skip proof

is easily established. Given $f$ continuous and $\varphi \in C^1$, using the fundamental theorem and the chain rule, we find

$$A'(t) = f(\varphi(t))\, \varphi'(t) = B'(t).$$

But $A(a) = B(a) = 0 \Rightarrow A(t) \equiv B(t)$

What is the analogue of (*) for multiple integrals?

Consider the case $n = 2$ and let $D$ be a "nice" bounded open region in the $x, y$ plane (perhaps elementary) and similarly $D^*$ a "nice" bounded region in the $u, v$ plane.

Further let $T: D^* \to D$ be of class $C^1$ (continuously differentiable map) which is one to one and onto ($D = T(D^*)$)

so $T(u,v) = (x(u,v), y(u,v))$

Then for any integrable (say continuous) function on $D$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Here $\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = x_u y_v - x_v y_u$

$= \det(DT(u,v))$ is the

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Jacobian determinant of the mapping $T$ and $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ is its <u>absolute value</u>

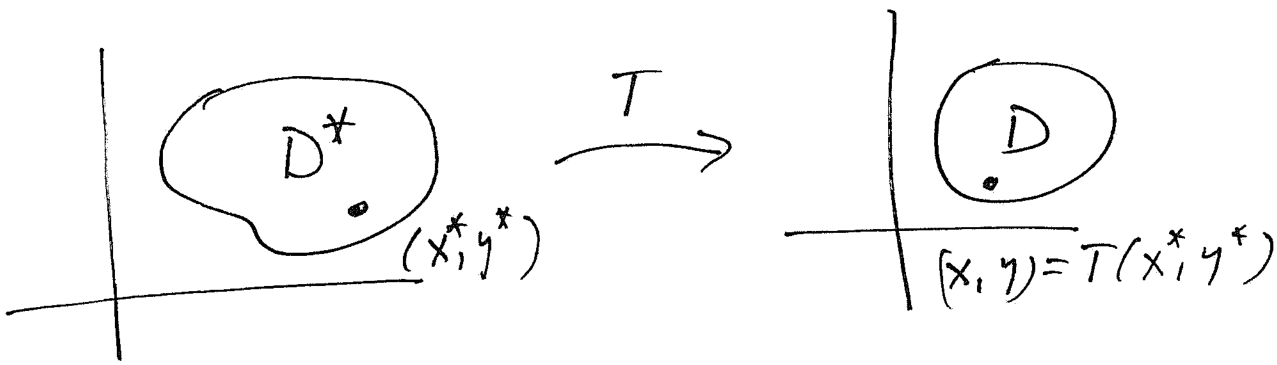
The proof of this theorem will be sketched next lecture. For the moment, we focus on the geometry of maps from $\mathbb{R}^2 \to \mathbb{R}^2$

6.1

Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

Let $T : D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$

be a nice map ($C^1$)




<u>Example</u> 1 Let $D^* = [0,1] \times [0, 2\pi]$ be a rectangle in $(r, \theta)$ space,

$0 \le r \le 1$, $0 \le \theta \le 2\pi$ (polar coord. space). Let $T$ be the polar coordinate "change of variables"

$$T(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$$

where $(x,y)$ are cartesian coordinates

Since $x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$,

the image $T(D^*) \subset$ unit disk

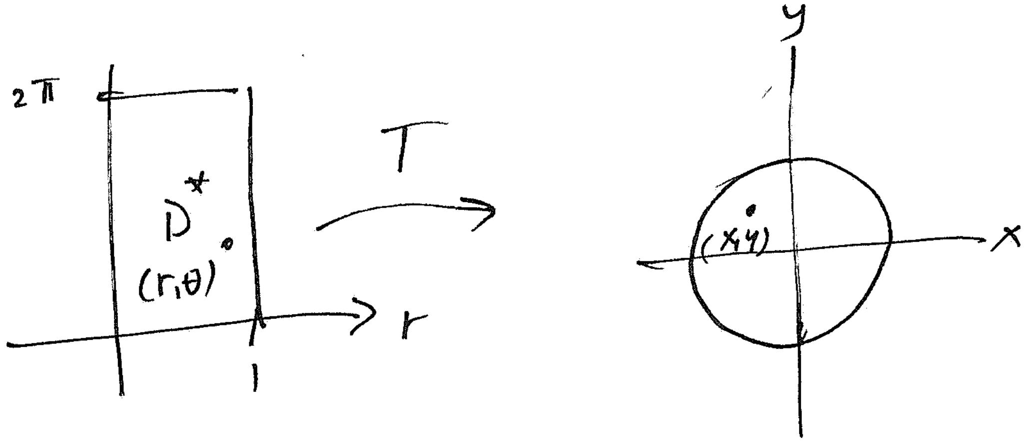
(since $0 \le r \le 1$). On the other hand every pt $(x,y)$ in the unit disk can be written as

$x = r\cos\theta$, $y = r\sin\theta$ for some

$0 \le r \le 1$, $0 \le \theta \le 2\pi$ so

the map $T$ is onto the unit disk.

Note $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$




**Example 2** Let $T(x,y) = (u,v) = \left( \frac{x+y}{2}, \frac{x-y}{2} \right)$

$D^* = [-1,1] \times [-1,1] \subset \mathbb{R}^2$, the square of side length 2 centered at $(0,0)$

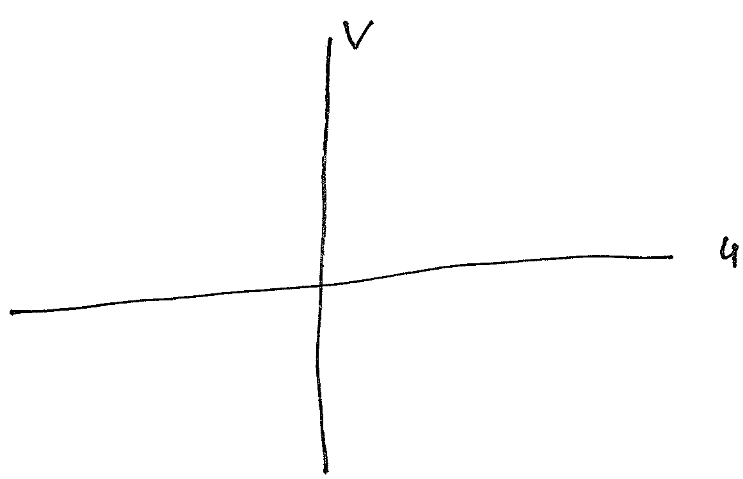
what is $D = T(D^*)$ ?

**Remark** Note that $T$ is a linear map from $\mathbb{R}^2$ with coordinates $(x,y)$ to $\mathbb{R}^2$ with

with coordinates $(u,v)$

so is represented by the $2 \times 2$ matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ (i.e $T(x,y) = A\binom{x}{y}$ written as a column vector)

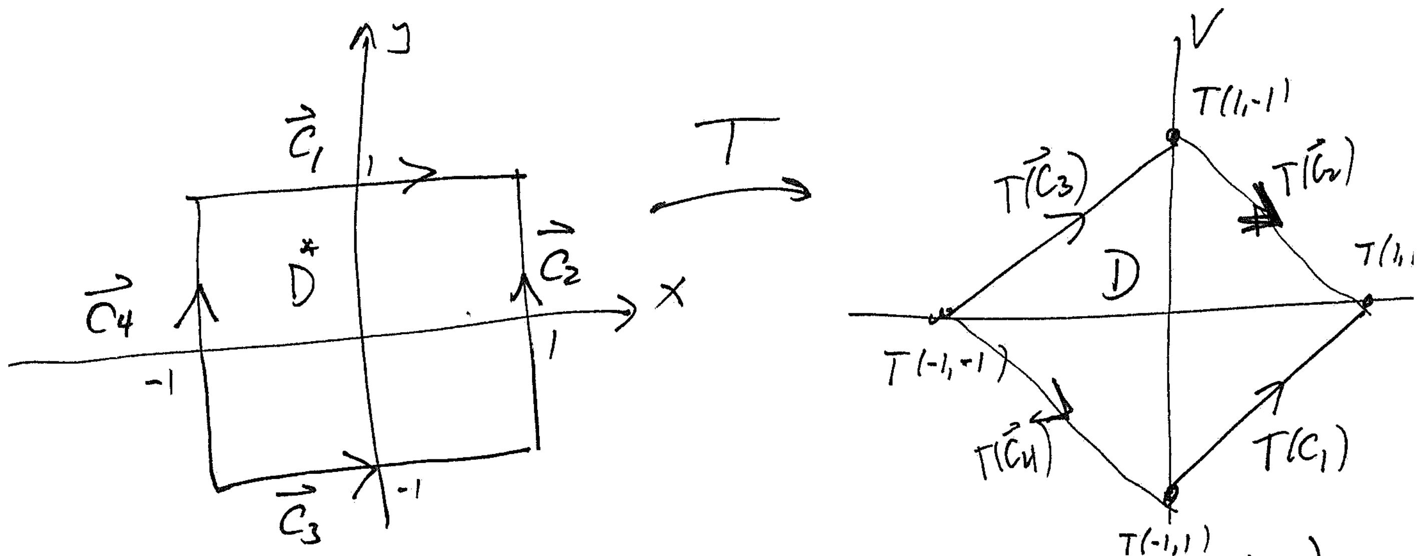
The first column of $A$ is $T\vec{\imath} = T(1,0)$. The second column is $T\vec{\jmath} = T(0,1)$. (Note also that because $T$ is linear $A = JT$ is constant)



## Example 2 (continued)

Let $T(x,y) = \left( \frac{x+y}{2}, \frac{x-y}{2} \right) = (u,v)$

$$D^* = [-1,1] \times [-1,1]$$




$$T(\vec{c}_1(t)) = \left( \frac{t+1}{2}, \frac{t-1}{2} \right)$$

$$-1 \le t \le 1$$

$\vec{c}_1(t) = (t,1), \ -1 \le t \le 1$

$\vec{c}_2(t) = (1,t), \ -1 \le t \le 1$

$\vec{c}_3(t) = (t,-1), \ -1 \le t \le 1$

$\vec{c}_4(t) = (-1,t), \ -1 \le t \le 1$

parametrization ?

$$v - u = -1$$

$$T(\vec{c}_3(t)) = \left( \frac{t-1}{2}, \frac{t+1}{2} \right)$$

# Linear and affine maps from $\mathbb{R}^2 \to \mathbb{R}^2$

**Defn** We say $T$ is linear if

$$T(\vec{x}_1 + \vec{x}_2) = T\vec{x}_1 + T\vec{x}_2$$

$$T(c\vec{x}) = c\,T(\vec{x}) \qquad c \in \mathbb{R}$$

Thus if $T$ is defined on $\mathbb{R}^2$ with coordinates $(x_1, x_2)$, we can represent $T$ by a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $T(x_1, x_2) = (a x_1 + b x_2,\ c x_1 + d x_2)$

$$= \left(\text{Transpose } ?\right) \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(Note the columns of $A$ are $T\vec{e}_1$ and $T\vec{e}_2$)

$\vec{e}_1 = \langle 1,0 \rangle$, $\vec{e}_2 = \langle 0,1 \rangle$

An affine map $S$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is of the form $S(x_1, x_2) = T(x_1, x_2) + (c_1, c_2)$

i.e the composition of $T$ with a translation by $(c_1, c_2)$. Thus we need only study $T$.

# Important properties of linear maps

1. T linear maps lines to lines.

Let $\vec{c}(t) = \vec{x}_0 + t\vec{v}$. Then

$T(\vec{c}(t)) = T(\vec{x}_0) + t\,T\vec{v}$ so

the image of $\vec{c}(t)$ is a line through $T(\vec{x}_0)$ in the direction $T\vec{v}$

~~2. T is one to one~~

~~i.e if $T\vec{x}_1 = T\vec{x}_2$ then~~

~~$\vec{x}_1 = \vec{x}_2$~~

2. Suppose the columns of $A$ are linearly independent, i.e. assume $\det A \neq 0$ ($\Longleftrightarrow$ $T\vec{e}_1, T\vec{e}_2$ linearly independent)

Then $T$ is 1-1, i.e. $T\vec{x}_1 = T\vec{x}_2 \Longrightarrow \vec{x}_1 = \vec{x}_2$

For since $T$ is linear, $T(\underbrace{\vec{x}_1 - \vec{x}_2}_{\vec{y} = (y_1, y_2)}) = 0$

$$\Longleftrightarrow \begin{array}{l} a y_1 + b y_2 = 0 \\ c y_1 + d y_2 = 0 \end{array} \Longleftrightarrow y_1 = y_2 = 0$$

i.e. only trivial solutions of the homogeneous linear equations.

3. again assuming $\det A \neq 0$

$T$ is onto, that is

given $(u,v)$ $\exists!$ $x,y$ s. that

$T(x,y) = (u,v)$ $\Longleftrightarrow$

$$\begin{aligned} ax + by &= u \\ cx + dy &= v \end{aligned} \qquad ad-bc \neq 0$$

$\Rightarrow$ unique soln $(x,y)$

$$x = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{ad-bc}, \quad y = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{ad-bc}$$

(Cramer's rule)

In the previous example 2

~~we~~ $$u = \frac{x+y}{2}$$

$$v = \frac{x-y}{2}$$

we can invert $T$

$$x = u+v$$

$$y = u-v$$

Again assuming $\det A \neq 0$

4. $T$ maps parallelograms to parallelograms.

A parallelogram in the $x, y$ plane is of the form $\alpha \vec{v} + \beta \vec{w} + \vec{p}$

$0 \leq \alpha, \beta \leq 1$

$\vec{v}, \vec{w}$ linearly independent

so $T(\alpha \vec{v} + \beta \vec{w} + \vec{p})$

$= \alpha T\vec{v} + \beta T\vec{w} + T(\vec{p})$

is again a parallelogram since

$\det A \neq 0 \Rightarrow T\vec{v}, T\vec{w}$ linearly indep't

Example (degenerate linear)

$$T(x,y) = (x,0) = (u,v)$$

$$T\vec{e}_1 = (1,0) \quad T(\vec{e}_2) = (0,0)$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \det A = 0$$

T maps all of $R^2$ onto the u axis of the image plane

Remark $\quad T(x,y) = (ax+by,\ cx+dy) = (u,v)$

the Jacobian

$$JT = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

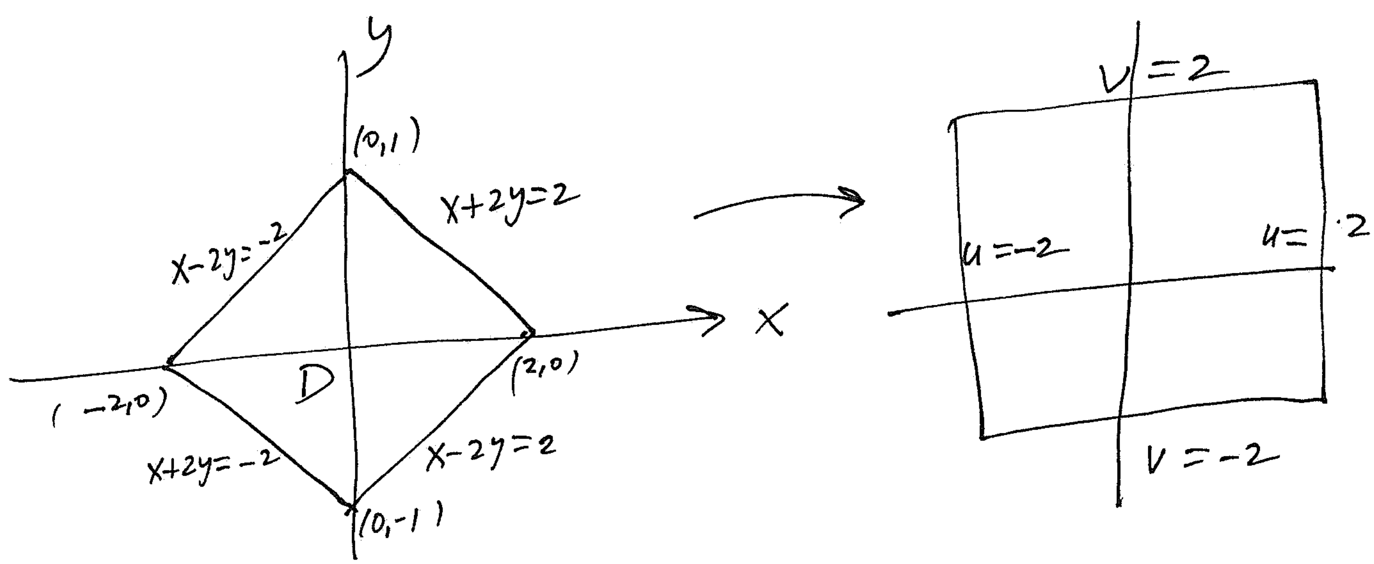
constant

<u>Example</u> Map the region $D$ bounded by the lines

$x+2y=2$, $x-2y=2$

$x+2y=-2$, $x-2y=-2$

onto a square $D^*$ centered at $(0,0)$ in the $u,v$ plane.



It is natural to try the linear transformation

$$u = x + 2y$$
$$v = x - 2y$$

which maps the parallelogram $D$ 1-1, onto the square $D^*$ $-2 \le u \le 2$, $-2 \le v \le 2$

Solving for the inverse transformation,

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{4}$$

Then $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} = -\frac{1}{4}$$

According to the change of variable theorem stated at the beginning, we can evaluate $\iint_D (3x+6y)^2 \, dA$

$$= \iint_{D^*} \left(3\left(\frac{u+v}{2}\right) + 6\left(\frac{u-v}{4}\right)\right)^2 \cdot \left|-\frac{1}{4}\right| du\, dv$$

$$= \int_{-2}^{2}\int_{-2}^{2} 9u^2 \cdot \frac{1}{4} \, du\, dv \qquad (= 48)$$

<u>Remark</u> You can evaluate the original integral directly but it is much more work.

# The change of variable formula for double integrals

The change of variables formula from Calculus 1 says:

$$(*) \quad A(t) = \int_a^t f(\varphi(x))\,\varphi'(x)\,dx =$$

$$\int_{\varphi(a)}^{\varphi(t)} f(x)\,dx \;:=\; B(t)$$

Here $f$ is continuous and $\varphi \in C^1$

By the Fundamental theorem of Calculus and the Chain Rule, fund. thm $\qquad\qquad\qquad\qquad$ chain rule

$$A'(t) \overset{\swarrow}{=} f(\varphi(t))\,\varphi'(t) \overset{\swarrow}{=} B'(t)$$

But $A(a) = B(a) = 0$ so

$$A(t) \equiv B(t)$$

What is the analogue of (*) for multiple integrals?

Consider the case $n=2$. Let $D$ be a "nice" bounded open region in the $x,y$ plane

Similarly let $D^*$ be a "nice" bounded open region in the $u, v$ plane. Further, let $T : D^* \to D$ be a $C^1$ (continuously differentiable) mapping which is 1-1 and onto (so $D = T(D^*)$)

Write $T(u,v) = (x(u,v), y(u,v))$

and $\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$

$= x_u y_v - x_v y_u .$

Then for any integrable $f: D \to \mathbb{R}$, we have

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

<u>Example 1</u> The polar coordinate map

$x = r\cos\theta \quad y = r\sin\theta$ on

$$D^* = [0,1] \times [0, 2\pi]$$

$$T(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$$

and $D = T(D^*) = \{(x,y) : x^2+y^2 \le 1\}$

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Then $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta) \cdot r\,dr\,d\theta$

If $f \equiv 1$ $A(D) = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \pi$

<u>Example 2</u> The density of an object occupying the upper hemisphere $W = \{(x,y,z) : x^2+y^2+z^2 \leq 4, z \geq 0\}$ is $\rho(x,y,z) = z$. Find the total mass of the object.

$$M = \iiint_W z\,dV = \iint_D \int_0^{\sqrt{1-x^2-y^2}} z\,dz\,dA$$

where $D = \{(x,y) : x^2+y^2 \leq 1\}$

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Then $M = \iint_D \frac{1-x^2-y^2}{2}\, dA$

Introduce polar coordinates

as in example 1. Then

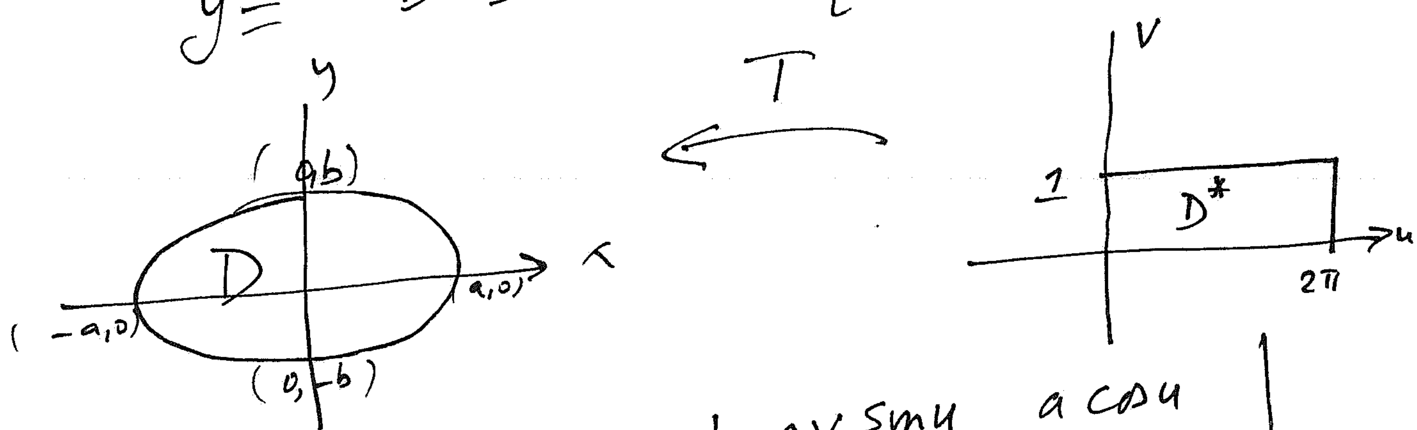
$$M = \iint_{D^*} \frac{1-r^2}{2} \cdot r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \frac{1-r^2}{2}\, r\, dr\, d\theta$$

$$= 2\pi \cdot \frac{1}{2} \int_0^1 (r-r^3)\, dr = \pi \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_{r=0}^{r=1}$$

$$= \frac{\pi}{4}$$

Example 4 Consider the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ in the $x,y$ plane. Introduce the change of variable

$$T: \begin{cases} x = av\cos u \\ y = bv\sin u \end{cases} \qquad D^* \begin{cases} 0 \le u \le 2\pi \\ 0 \le v \le 1 \end{cases}$$




Then $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -av\sin u & a\cos u \\ bv\cos u & b\sin u \end{vmatrix}$$

$$= -abv$$

Then $A(D) = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$

$= ab \int_0^{2\pi} \int_0^1 v\,dv\,du = ab \cdot 2\pi \cdot \frac{1}{2} = \pi ab$

Example 5 Evaluate $\iint_D (x+y)\,dA$

where $D$ is the region bounded by the lines $y = x$, $y = 3x$ and $x + y = 4$

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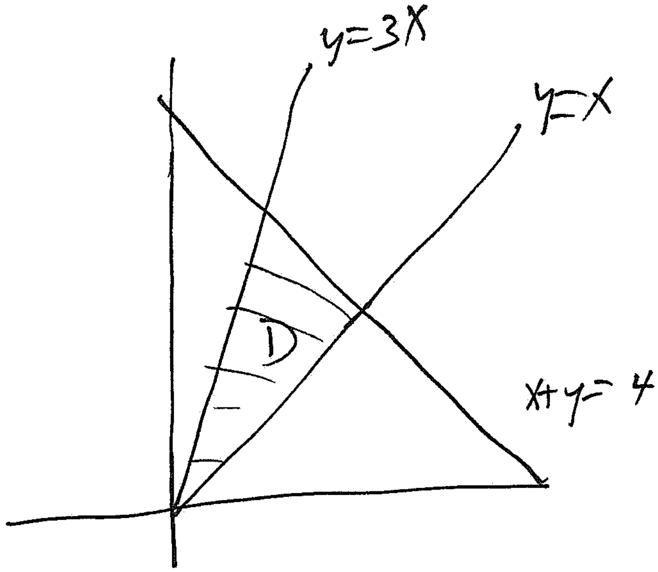



Let $T(u,v) = \left(\frac{u-v}{2}, \frac{u+v}{2}\right) = (x,y)$

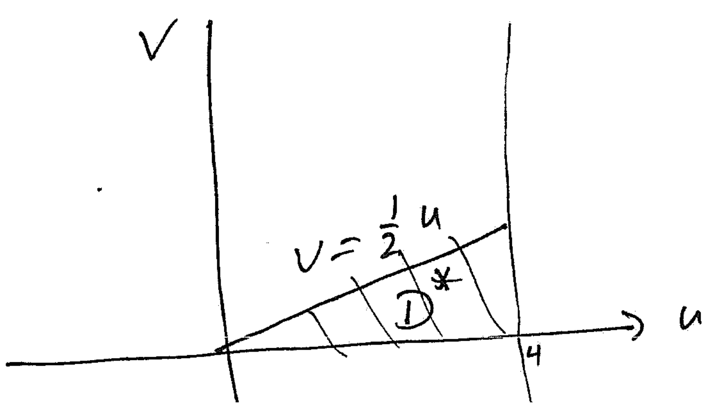
Then $u = x+y$
$v = y-x$

$y=3x \longleftrightarrow \frac{u+v}{2} = \frac{3}{2}(u-v)$

$\longleftrightarrow u = 2v$ n $v = \frac{1}{2}u$

$y = x \longleftrightarrow v = 0$

$x+y=4 \longleftrightarrow u = 4$

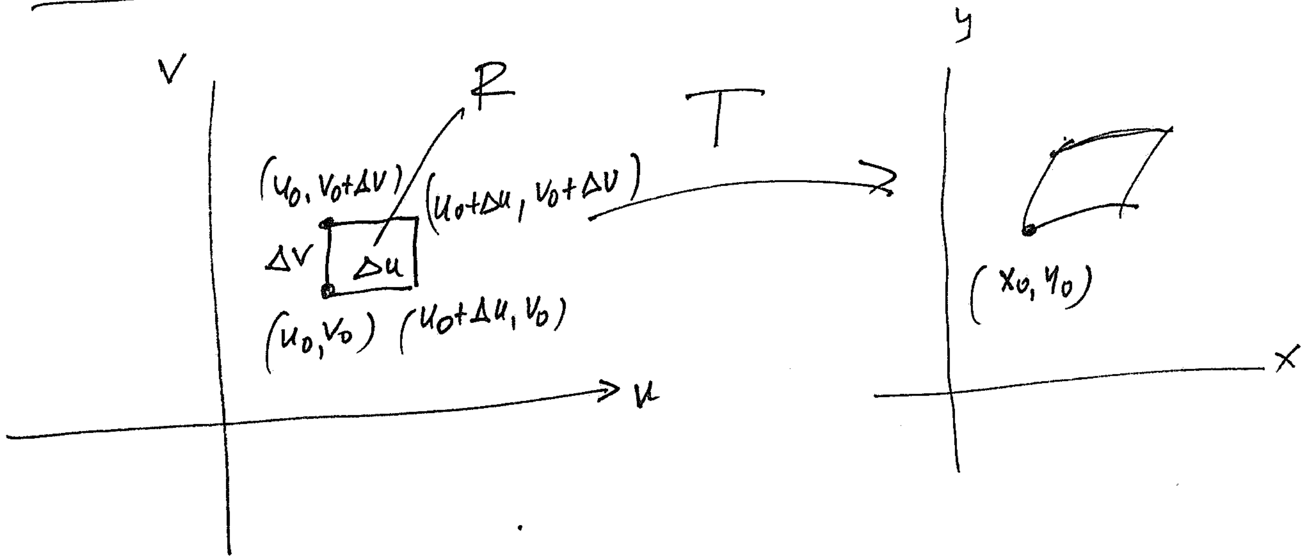


$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$$

$$\iint_D (x+y)\, dA = \iint_{D^*} u \cdot \frac{1}{2}\, du\, dv$$

$$= \int_0^4 \int_0^{\frac{1}{2}u} \frac{u}{2}\, dv\, du = \int_0^4 \frac{u^2}{4} du = \frac{4^3}{3 \cdot 4} = \frac{16}{3}$$

# Idea of the proof




Since $T$ is differentiable

$$T(u,v) = T(u_0, v_0) + dT(u_0, v_0)\begin{pmatrix}\Delta u \\ \Delta v\end{pmatrix} + \text{smaller error}$$

$$dT(u_0,v_0) = \begin{pmatrix} x_u(u_0,v_0) & x_v(u_0,v_0) \\ y_u(u_0,v_0) & y_v(u_0,v_0) \end{pmatrix}$$

so the 4 corners of the small rectangle $R$

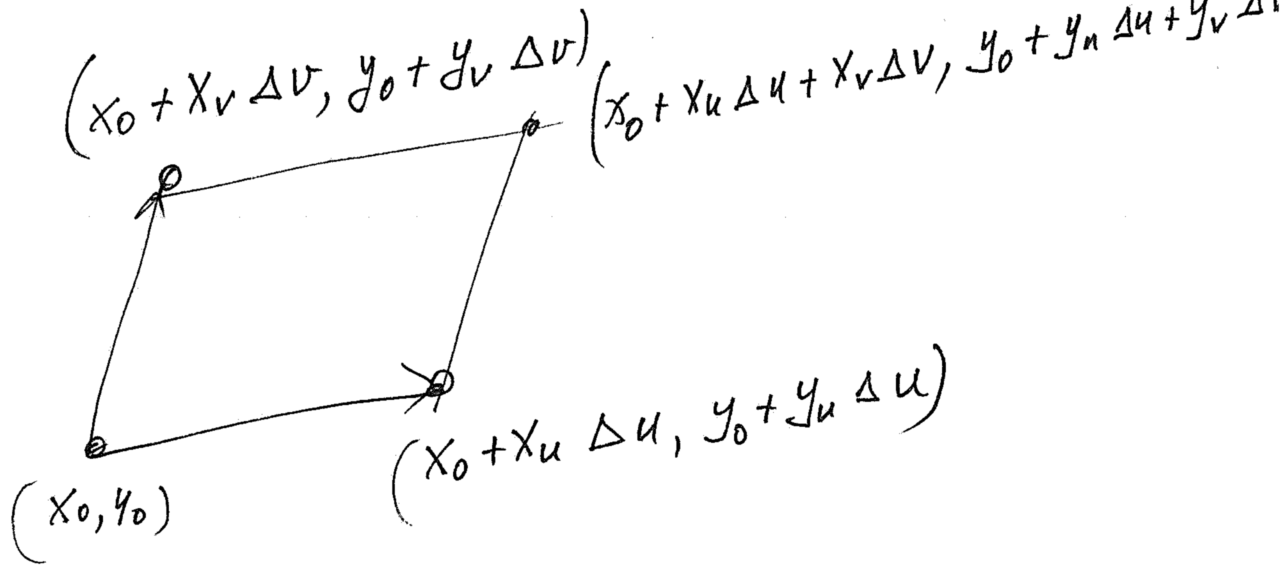
are mapped to (approximately)

so

$$X(u,v) = x_0 + X_u(u_0,v_0)(u-u_0) + X_v(u_0,v_0)(v-v_0) + \text{smaller error}$$

$$y(u,v) = y_0 + y_u(u_0,v_0)(u-u_0) + y_v(u_0,v_0)(v-v_0) + \text{smaller error}$$

so the "4 corners" of the parallelogram in the x, y plane are



so the area of the image $T(R)$

is (to first order)

$$\left| (x_u(u_0,v_0), y_u(u_0,v_0))\Delta u \times (x_v(u_0,v_0), y_v(u_0,v_0))\Delta v \right|$$

$$= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \, \Delta v$$

(so $dx\,dy \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$

Summing over rectangles in the Riemann sum makes the formula

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

plausible

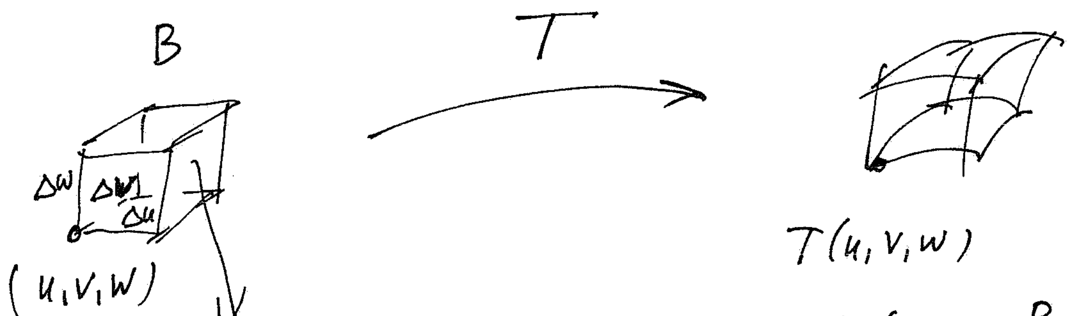
# Change of variables in triple integrals

The idea behind the change of variables theorem in three variables

$$(u,v,w) \overset{T}{\longmapsto} (x,y,z)$$

is analogous to two variables (only more complicated!)

How does $T$ change the volume of an infinitesimal box $B$?

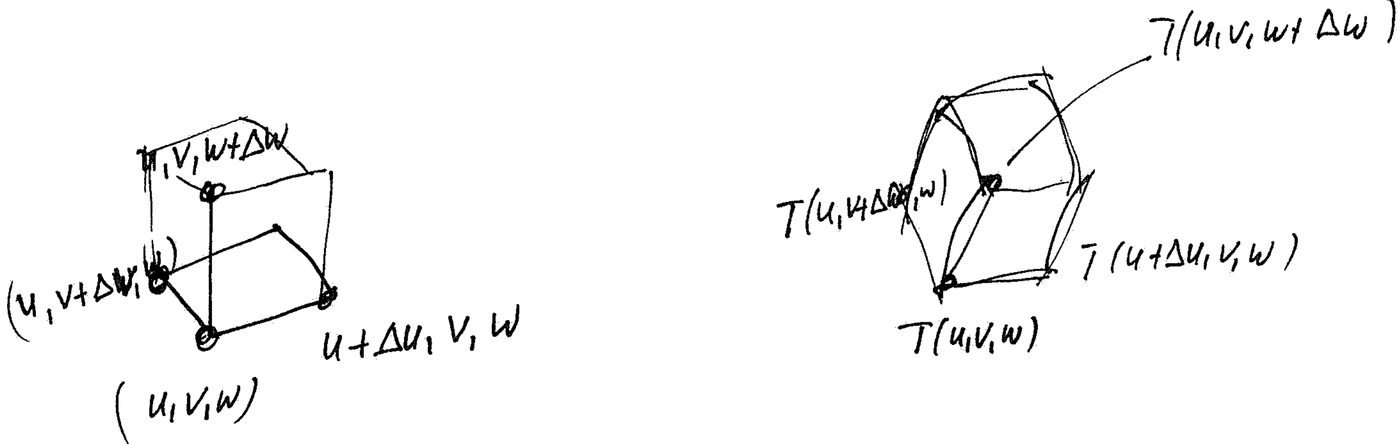



volume $\Delta u \Delta v \Delta w$

$T$ deforms $B$ to a "curvy box $T(B)$"

Since $T$ is differentiable, $T$ is well approximated by its linear approximation (differential)

so $T(B) \approx$ infinitesimal parallelpiped




The image $T(B)$ is generated by the three vectors

$$T(u+\Delta u, v, w) - T(u,v,w) \approx T_u(u,v,w)\,\Delta u$$
$$T(u, v+\Delta v, w) - T(u,v,w) \approx T_v(u,v,w)\,\Delta v$$
$$T(u, v, w+\Delta w) - T(u,v,w) \approx T_w(u,v,w)\,\Delta w$$

The volume of the curvy $T(B)$ is approximately given by the scalar triple product

$$\Delta V \approx \left| (T_u \Delta u \times T_v \Delta v) \cdot T_w \Delta w \right|$$
$$= \left| (T_u \times T_v) \cdot T_w \right| \Delta u \Delta v \Delta w$$
$$\approx \underbrace{\left| \det DT(u,v,w) \right|}_{= JT(u,v,w)} \Delta u \Delta v \Delta w$$

$JT(u,v,w)$ the Jacobian matrix of the map $T$

<u>Notation</u> Write $T(u,v,w)$

$$= \big(x(u,v,w), y(u,v,w), z(u,v,w)\big)$$

Then we write

$$\det JT(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

So the terms of the Riemann sum for the triple integral $\iiint_W f(x,y,z)\,dV$ have the form

$$f\big(T(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \Delta u\,\Delta v\,\Delta w$$

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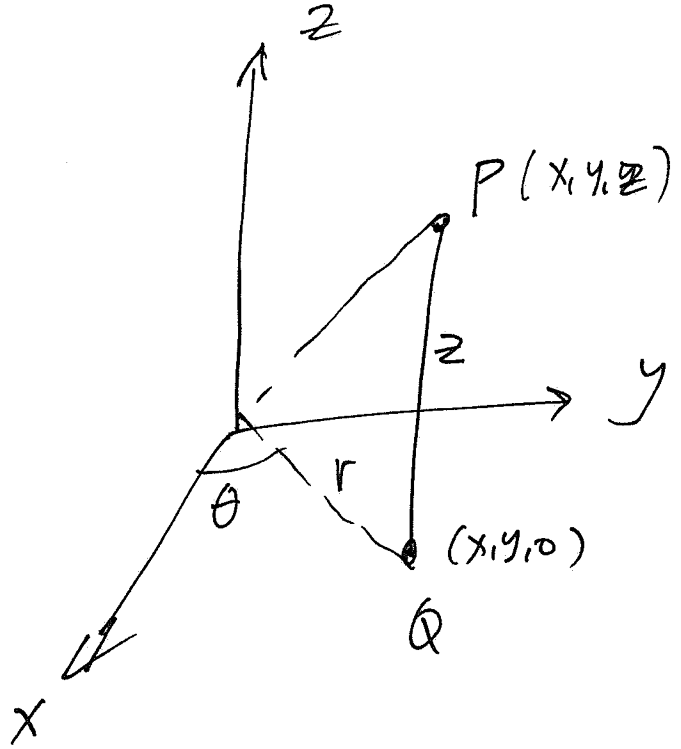
In the limit as $\Delta u, \Delta v, \Delta w \to 0$:

$$\iiint_W f(x,y,z)\,dV = \iiint_{W^* = T(W)} f(T(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

# cylindrical coordinates




use polar coordinates for Q

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Then $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$

<u>Example</u> 1. Let $W$ be the region bounded above by $z = 16 - x^2 - y^2$ and below by the $x, y$ plane.

Evaluate $\iiint\limits_W (8 + x + y)\, dV =$

Introduce cylindrical coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \qquad \left.\begin{aligned} 0 &\le z \le 16 - r^2 \\ 0 &\le r \le 4 \\ 0 &\le \theta \le 2\pi \end{aligned}\right\} W^*$$

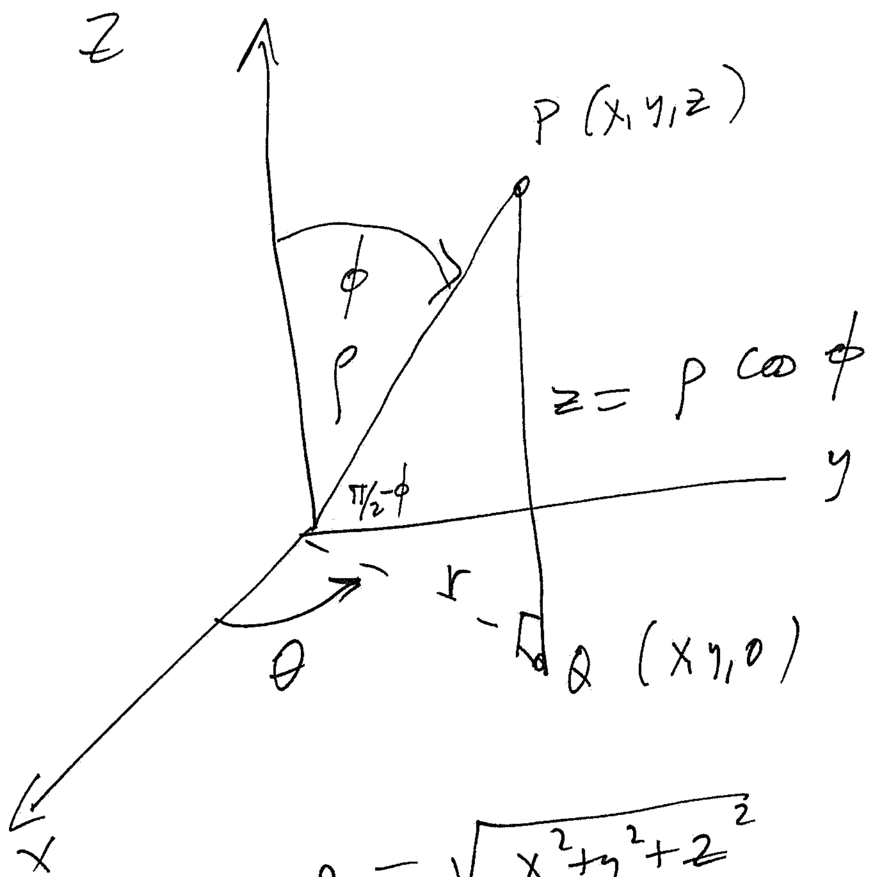
$$= \int_0^{2\pi}\int_0^4\int_0^{16-r^2} (8 + r\cos\theta + r\sin\theta)\, r\, dz\, dr\, d\theta$$

$$= \int_0^4\int_0^{2\pi}\int_0^{16-r^2} (8 + r\cos\theta + r\sin\theta)\, r\, dz\, d\theta\, dr$$

$$= \int_0^4\int_0^{2\pi} r(16 - r^2)(8 + r\cos\theta + r\sin\theta)\, d\theta\, dr$$

$$= \int_0^4 r(16-r^2)\cdot 16\pi \, dr = 1024\pi$$

## spherical coordinates



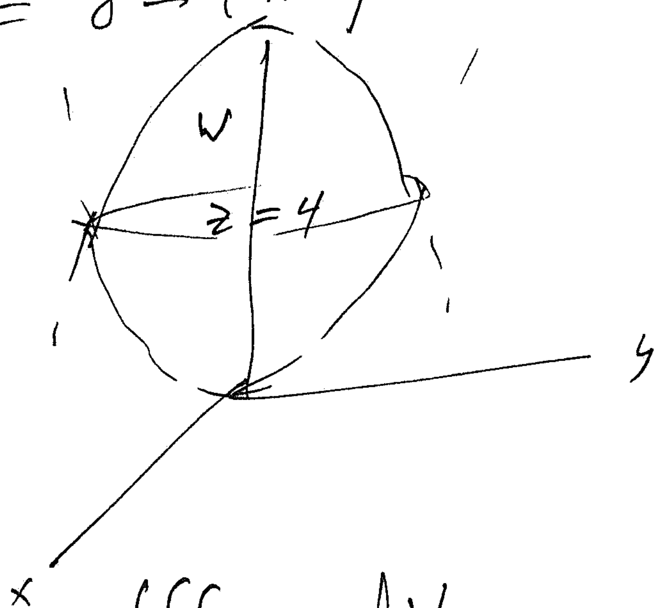

$\rho = \sqrt{x^2+y^2+z^2}$

$r = \sqrt{x^2+y^2}$
$= \rho \sin\phi$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

<u>Example</u> Let $W$ be the region enclosed by the paraboloids $z = x^2+y^2$ and $z = 8-(x^2+y^2)$




Evaluate $\iiint_W xyz \, dV$

$$= \int_0^{2\pi}\int_0^{2}\int_{r^2}^{8-r^2} (r\cos\theta)(r\sin\theta) z \cdot r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \left(\frac{(8-r^2)^2 - r^4}{2}\right) \cdot r^3 \, dr \, d\theta$$

$$= \pi \int_0^{2} (64-16r^2) r^3 \, dr$$

(skip calculation, just state) 9

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

Expand by cofactors along last row:

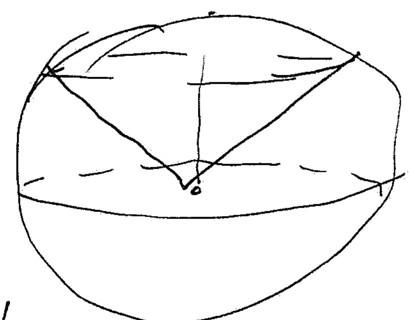
$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi \cdot -\rho^2\sin\phi\cos\phi - \rho\sin\phi\cdot\rho\sin^2\phi$$

$$= -\rho^2\sin\phi$$

$$\boxed{\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\underline{\rho^2\sin\phi\,d\rho\,d\theta\,d\phi}}$$

**Example 1** Find the volume of the ice cream cone region bounded above by the upper hemisphere $z = \sqrt{1-x^2-y^2}$ and below by the cone $z = x^2+y^2$



on the cone $z = \sqrt{x^2+y^2}$, $\phi = \pi/4$

$W^*$: $0 \le \theta \le 2\pi$, $0 \le \rho \le 1$, $0 \le \phi \le \pi/4$

$$\text{Vol } W = \int_0^1 \int_0^{2\pi} \int_0^{\pi/4} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \, d\phi = \frac{2\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right)$$

Example 2 The density of an object occupying the upper hemisphere of $x^2+y^2+z^2 \leq 1$ is $\rho(x,y,z) = z$. Find the total mass.

solution

We use spherical coordinates to evaluate $M = \iiint_W z\,dV$

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho\cos\phi \; \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{1}{4}\cdot 2\pi \int_0^{\pi/2} \sin\phi \, d(\sin\phi) = \frac{\pi}{4}$$

Remark

Do not forget you can use linear change of variable in Triple integrals.

Example Find the volume of the parallelpiped

$$0 \le 2x-3y+z \le 5$$
$$1 \le x+2y \le 4$$
$$-3 \le x-z \le 6$$

Introduce new variables

$$u = 2x-3y+z \qquad 0 \le u \le 5$$
$$v = x+2y \qquad 1 \le v \le 4$$
$$w = x-z \qquad -3 \le w \le 6$$

$W^*$

(B)

Then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix}$

cofactors along 3rd row

$= 1 \cdot \begin{vmatrix} -3 & 1 \\ 2 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}$

$= -2 - 7 = -9$

Jacobian of inverse

$\Rightarrow \left| \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \dfrac{1}{9}$

$\text{Vol} = \displaystyle\iiint_{W^*} \frac{1}{9}\, du\, dv\, dw$

$= \frac{1}{9} \cdot 5 \cdot 3 \cdot 9 = 15$