7-3 Parametrized surfaces

Defn a parametrization of a surface (patch)

is a mapping $X \in C^1$

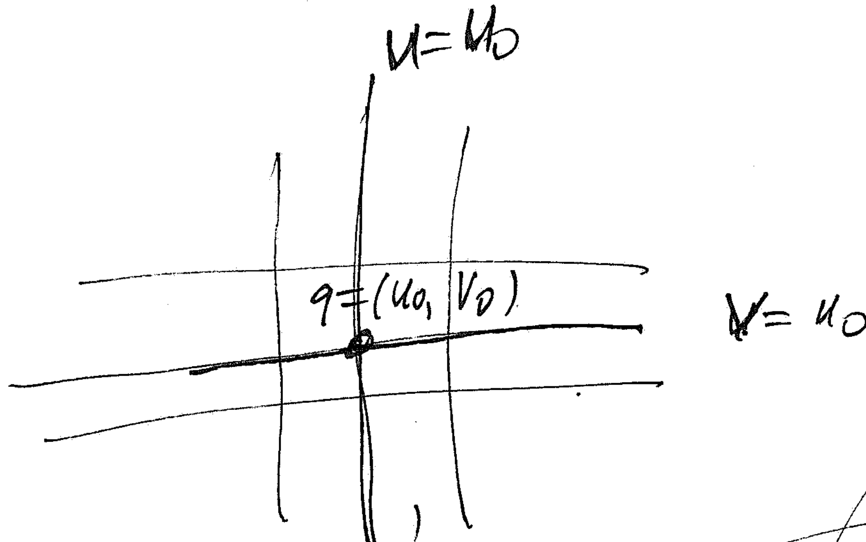
$X: D \subset R^2 \longrightarrow R^3$

$X(u,v) = (x(u,v), y(u,v), z(u,v))$

We require X to be 1-1 (with C^1 inverse)

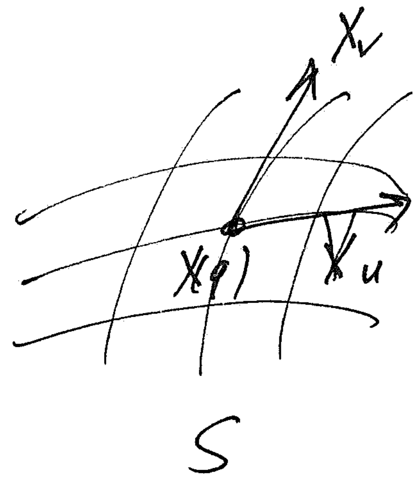
and X_u, X_v linearly indep't

i.e. $X_u \times X_v \neq 0$



$X = (x(u,v), y(u,v), z(u,v))$

u,v plane



$$dX_q(e_1) = X_u$$
$$dX_q(e_2) = X_v$$

$$JX(q) = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}(u_0, v_0)$$

rank 2

X_u, X_v linear indept span "tangent plane" to S, $\vec{N} = \dfrac{\vec{X}_u \times \vec{X}_v}{|X_u \times X_v|}$ = unit normal to S

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Ex 1 $X = \left(x, y, \sqrt{1-(x^2+y^2)}\right)$

$x, y \in U = \{(x,y) : x^2+y^2 < 1\}$

$$X_x = \left(1, 0, -\frac{x}{\sqrt{1-x^2-y^2}}\right)$$

$$X_y = \left(0, 1, -\frac{y}{\sqrt{1-x^2-y^2}}\right)$$

$$X_x \times X_y = \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right) \neq 0$$

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Ex 3 one sheeted cone

$$z = \sqrt{x^2+y^2}$$

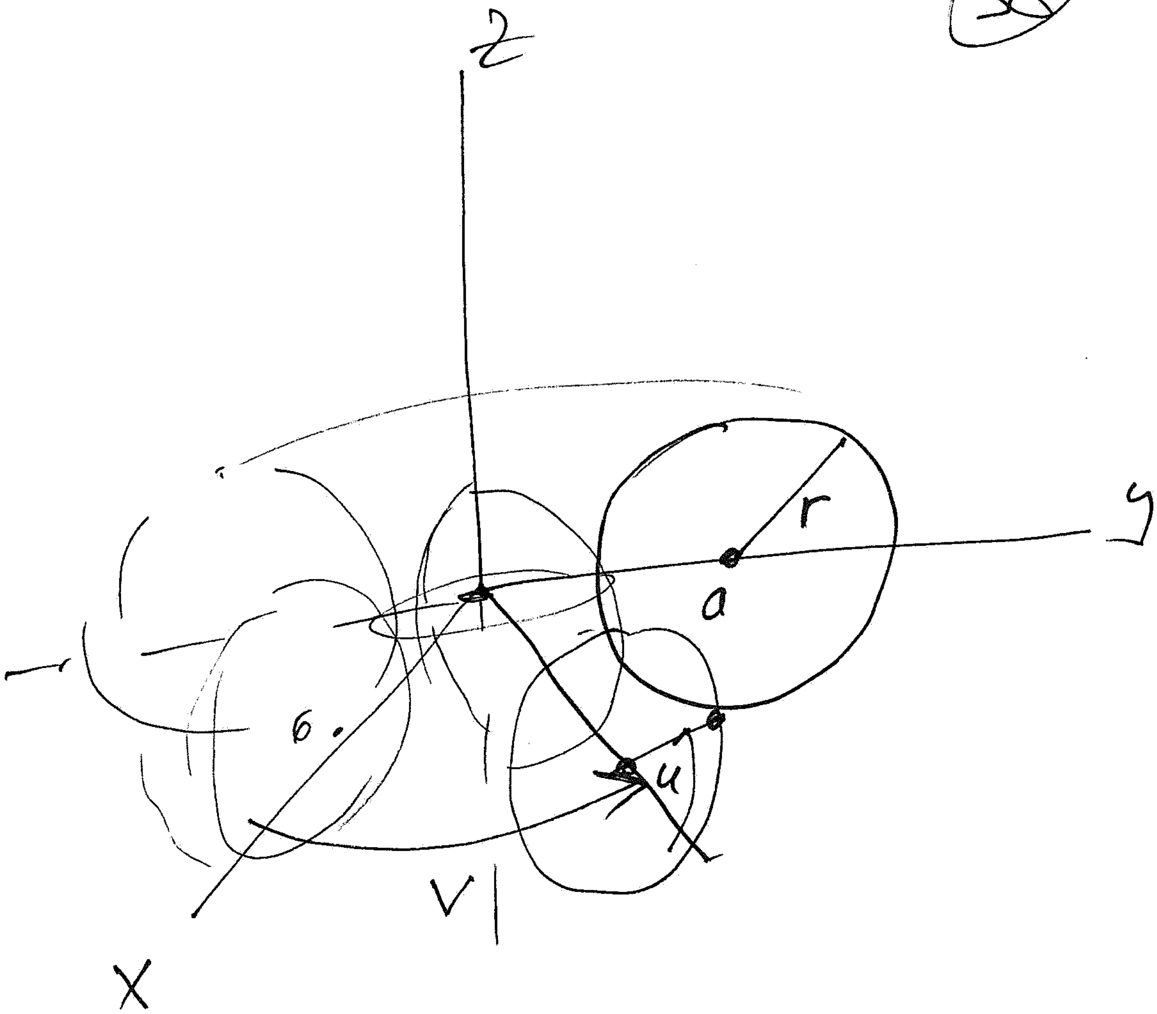
not regular at $(0,0,0)$

Ex 4 Torus of revolution

$$X(u,v) = \big((r\cos v + a)\cos u,\ (r\cos v + a)\sin u,\ r\sin v \big)$$

$$0 < u < 2\pi$$

$$0 < v < 2\pi$$


z
y
r
a
6
u
v
x

Ex 2

ellipsoid $S: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

is a regular surface

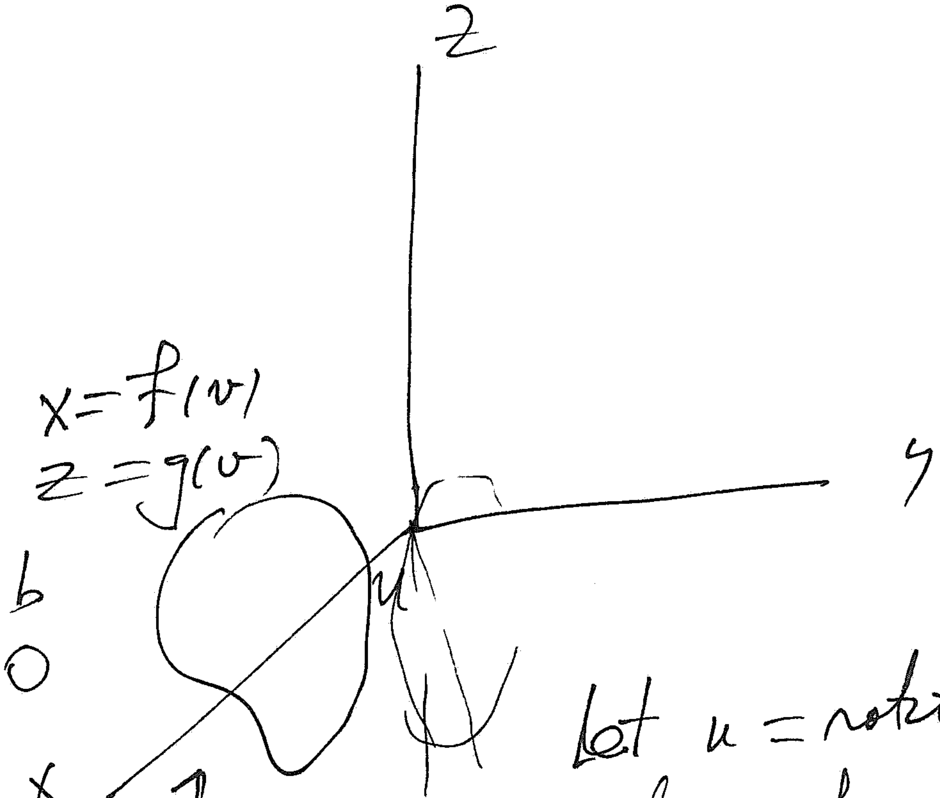
since it is $f^{-1}(0)$

$f = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$

and $\nabla f = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right)$

$\neq 0$ on S

Ex 5 Surface of revolution



$x = f(v)$
$z = g(v)$

$a < v < b$
$f(v) > 0$

Let u = rotation $\measuredangle$ about the z axis

plane curve in x, z plane

$$X(u,v) = \big(f(v)\cos u,\ f(v)\sin u,\ g(v)\big)$$

$$\begin{array}{cccc} & i & j & k \\ X_u = \langle & -f(v)\sin u, & f(v)\cos u, & 0 \ \rangle \\ X_v = \langle & f'(v)\cos u, & f'(v)\sin u, & g'(v) \ \rangle \end{array}$$

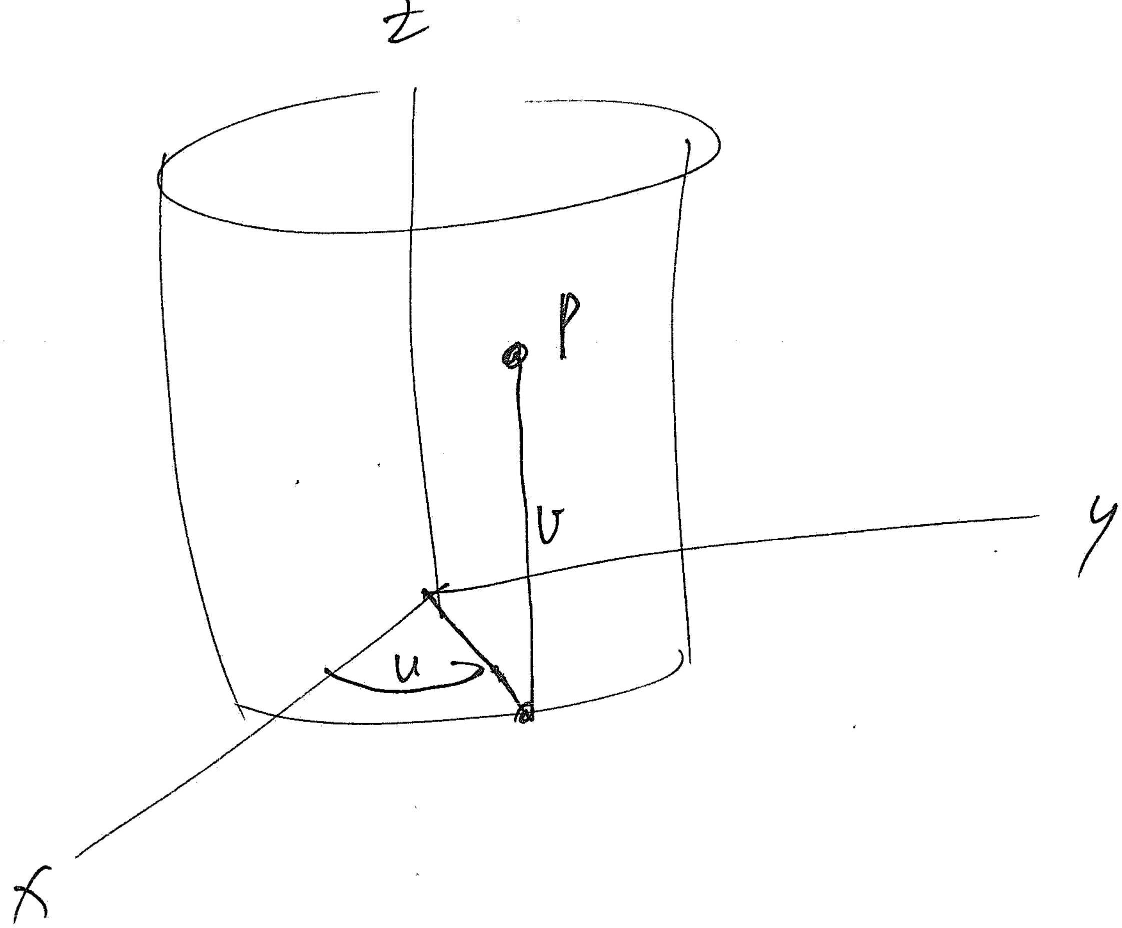
$$X_u \times X_v = \langle f(v)\,g'(v)\cos u,\ f(v)\,g'(v)\sin u,\ -f(v)\,f'(v) \rangle$$

Ex. cylinder of radius 1

$$X(u,v) = (\cos u, \sin u, v)$$

$$0 < u < 2\pi$$

$$-\infty < v < \infty$$



<u>Example</u> $X(u,v) = (u\cos v,\ u\sin v,\ u^2+v^2)$

$$\begin{array}{cccc} & i & j & k \\ X_u = (& \cos v, & \sin v, & 2u\ ) \\ X_v = (& -u\sin v, & u\cos v, & 2v\ ) \end{array}$$

$$X_u \times X_v = \langle 2v\sin v - 2u^2\cos v,\ -2v\cos^2 v - 2u\sin^2 v,\ u \rangle$$

$\cancel{|X_u \times X_v| = 0}$

$$|X_u \times X_v| = 4v^2\sin^2 v + 4u^4\cos^2 v - 8u^2 v \sin v \cos v$$
$$+ 4v^2\cos^4 v + 4u^2\sin^4 v$$
$$- 8uv\sin^2 v\cos^2 v$$
$$+ u^2$$

$X_u \times X_v = 0 \Rightarrow u = 0 \Rightarrow$

$X_u \times X_v = \langle 2v\sin v,\ 0,\ 0 \rangle$

$\Rightarrow v = 0$ or $\sin v = 0$

$v = 2\pi n \quad n \in \mathbb{Z}$

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at $(1,0)$

$$X(1,0) = (1,0,1)$$

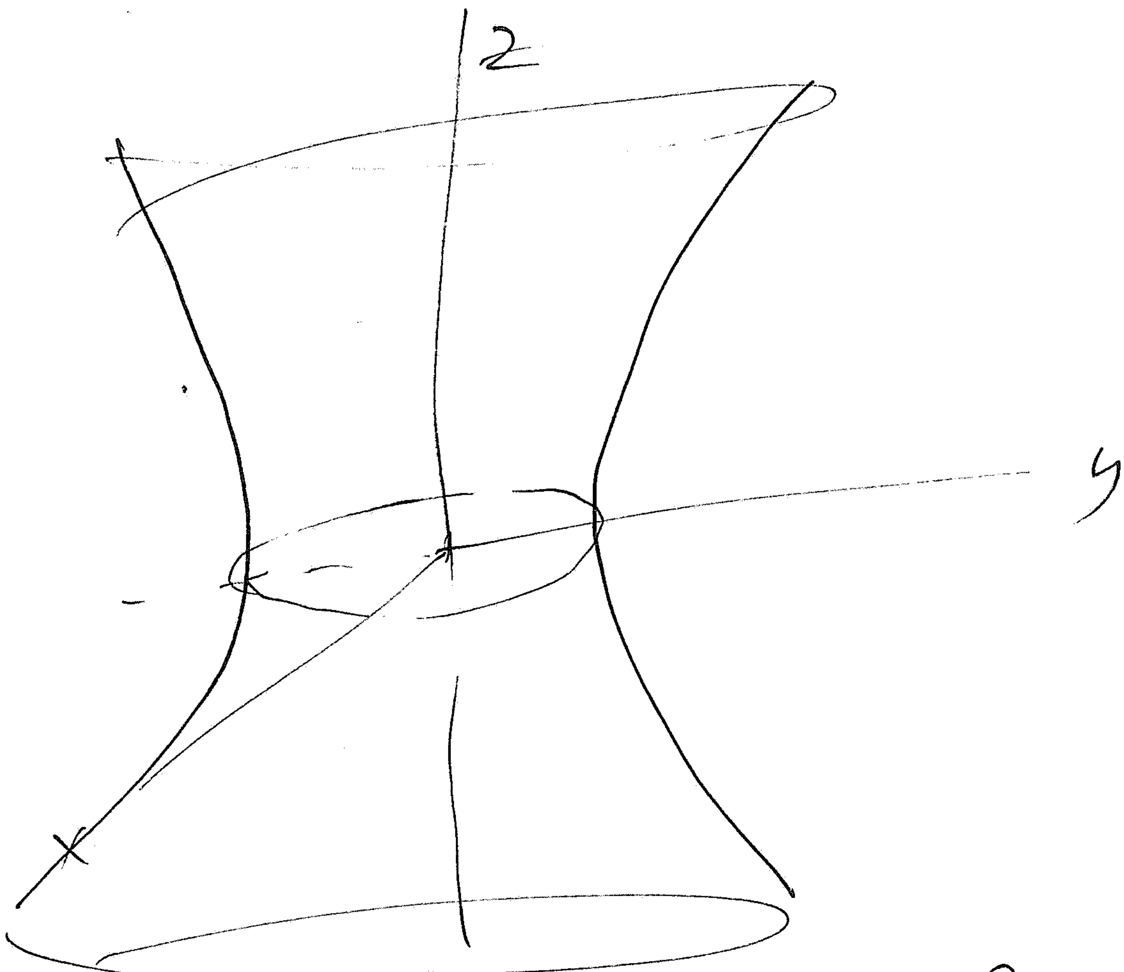
$$X_u \times X_v (1,0) = \langle -2, 0, 1 \rangle$$

tangent plane : $-2(x-1) + (z-1) = 0$

i.e $z = 2x - 1$

Ex hyperboloid of one sheet

$$x^2+y^2-z^2=1$$



$x = r\cos\theta \quad y = r\sin\theta$

$z^2 = r^2 - 1$ or

$r^2 - z^2 = 1$

$r = \cosh u\ ,\ z = \sinh u$

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$$X(u,\theta) = (\cosh u \cos\theta, \cosh u \sin\theta, \sinh u)$$

$$0 \le \theta \le 2\pi$$

$$-\infty < u < \infty$$

$$X_u = (\sinh u \cos\theta, \sinh u \sin\theta, \cosh u)$$

$$X_\theta = (-\cosh u \sin\theta, \cosh u \cos\theta, 0)$$

$$X_u \times X_\theta = \langle -\cosh^2 u \cos\theta, -\cosh^2 u \sin\theta, \cosh u \sinh u \rangle$$

$$|X_u \times X_\theta|^2 = \cosh^4 u + \cosh^2 u \sinh^2 u$$

$$= \cosh^2 u (\cosh^2 u + \sinh^2 u)$$

$$> 0$$

$\square$

Def'n Let D be an x-simple or y simple region in $\mathbb{R}^2$ and $X: D \to \mathbb{R}^3$ be a surface patch. We define

$$A(S) = \iint_D |\vec{X}_u \times \vec{X}_v| \, du\,dv$$

where $S = X(D)$

②

$$|\vec{X}_u \times \vec{X}_v| = \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

$$= \left(y_u z_v - y_v z_u,\ z_u x_v - z_v x_u,\ x_u y_v - x_v y_u \right)$$

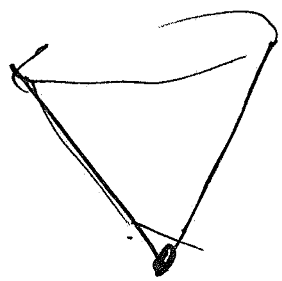
$$= \left(\frac{\partial(y,z)}{\partial(u,v)},\ \frac{\partial(z,x)}{\partial(u,v)},\ \frac{\partial(x,y)}{\partial(u,v)} \right)$$

$$A(S) = \iint_D \sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2}\, du\, dv$$

$$A(S) = \iint_U |X_u \times X_v| \, du\, dv$$

<u>Example</u> (cone) $z = \sqrt{x^2+y^2}$ $\quad x^2+y^2 \le 1$

graph $X = \left(x, y, \sqrt{x^2+y^2}\right)$



$$X_x = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right)$$

$$X_y = \left(0, 1, \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$X_x \times X_y = \left(-\frac{x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right)$$

singular at $(0,0)$

$$|X_x \times X_y| = \sqrt{2}$$

$$A = \lim_{\varepsilon \to 0} \iint_{B_1 - B_\varepsilon} \sqrt{2}\, dx\, dy = \pi \sqrt{2}$$

area of a graph $z = f(x, y)$

$$X = (x, y, f(x, y))$$

$$X_x = (1, 0, f_x)$$

$$X_y = (0, 1, f_y)$$

$$X_x \times X_y = \langle -f_x, -f_y, 1 \rangle$$

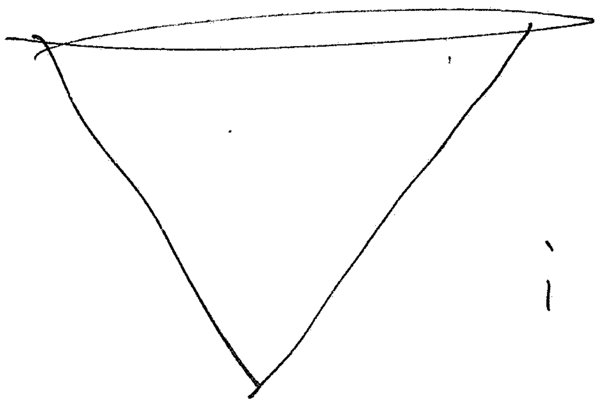
$$|X_x \times X_y| = \sqrt{1 + |Df|^2}$$

$$A = \iint_D \sqrt{1 + |Df|^2}\, dx\, dy.$$

Ex 1 (cone)

$X(r,\theta) = (r\cos\theta, r\sin\theta, r)$

$z = \sqrt{x^2+y^2}$



D: $0 < r \le 1$
$0 \le \theta \le 2\pi$

$i \quad j \quad k$

$X_r = (\cos\theta, \sin\theta, 1)$

$X_\theta = (-r\sin\theta, r\cos\theta, 0$

$X_r \times X_\theta = \langle -r\cos\theta, -r\sin\theta, r \rangle$

$|X_r \times X_\theta| = r\sqrt{2}$

$$A(S) = \int_0^{2\pi}\int_0^1 \sqrt{2}\, r \cdot dr\, d\theta$$

$$= \int_0^{2\pi} \frac{\sqrt{2}}{2}\, d\theta = \sqrt{2}\,\pi$$

Ex 2 (Helicoid)

$x = r\cos\theta \quad y = r\sin\theta \quad z = \theta$

$D: \quad 0 \le \theta \le 2\pi$

$\quad\quad 0 \le r \le 1$

$X(r,\theta) = (r\cos\theta,\ r\sin\theta,\ \theta)$

$$\begin{array}{cccc} & i & j & k \\ X_r = (& \cos\theta, & \sin\theta, & 0) \\ X_\theta = (& -r\sin\theta, & r\cos\theta, & 1) \end{array}$$

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$$\vec{X}_r \times \vec{X}_\theta = \langle \sin\theta, -\cos\theta, r \rangle$$

$$|\vec{X}_r \times \vec{X}_\theta| = \sqrt{1+r^2}$$

$$A = \iint_D \sqrt{r^2+1}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\, dr\, d\theta$$

$$= 2\pi \int_0^1 \sqrt{r^2+1}\, dr$$

$r = \tan\theta$ $\quad = 2\pi \int_0^{\pi/4} \sec\theta \sec^2\theta\, d\theta$

$$= \pi\left(\sqrt{2} + \log(1+\sqrt{2})\right)$$

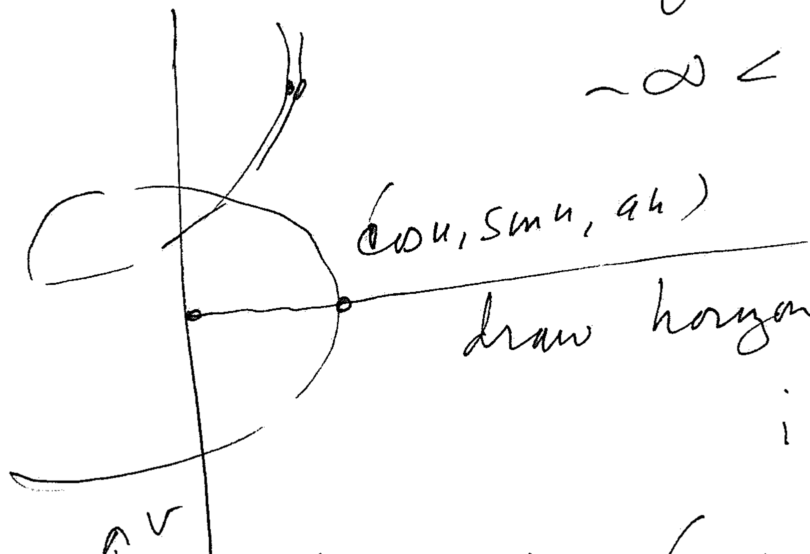
(8)

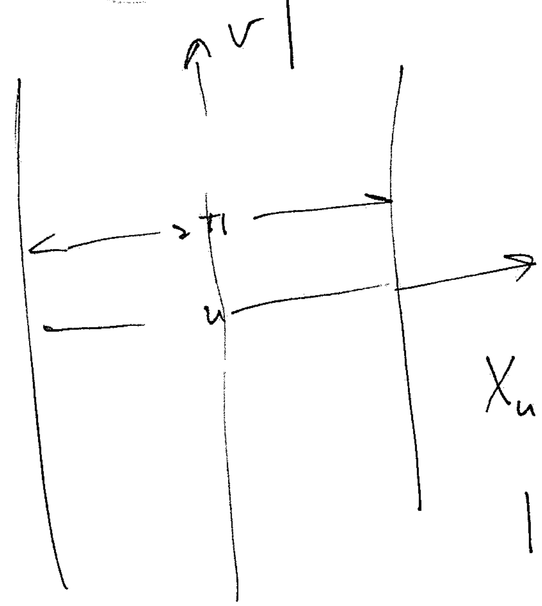
Ex (helicoid) helix $(\cos u, \sin u, au)$

$$X(u,v) = (v\cos u, v\sin u, au)$$

$$0 < u < 2\pi$$

$$-\infty < v < \infty$$





$$\begin{array}{cccc} & i & j & k \\ X_u = (& -v\sin u, & v\cos u, & a\ ) \\ X_v = (& \cos u, & \sin u, & 0\ ) \end{array}$$

$$X_u \times X_v = (-a\sin u, a\cos u, -v)$$

$$|X_u \times X_v| = \sqrt{a^2+v^2} > 0$$

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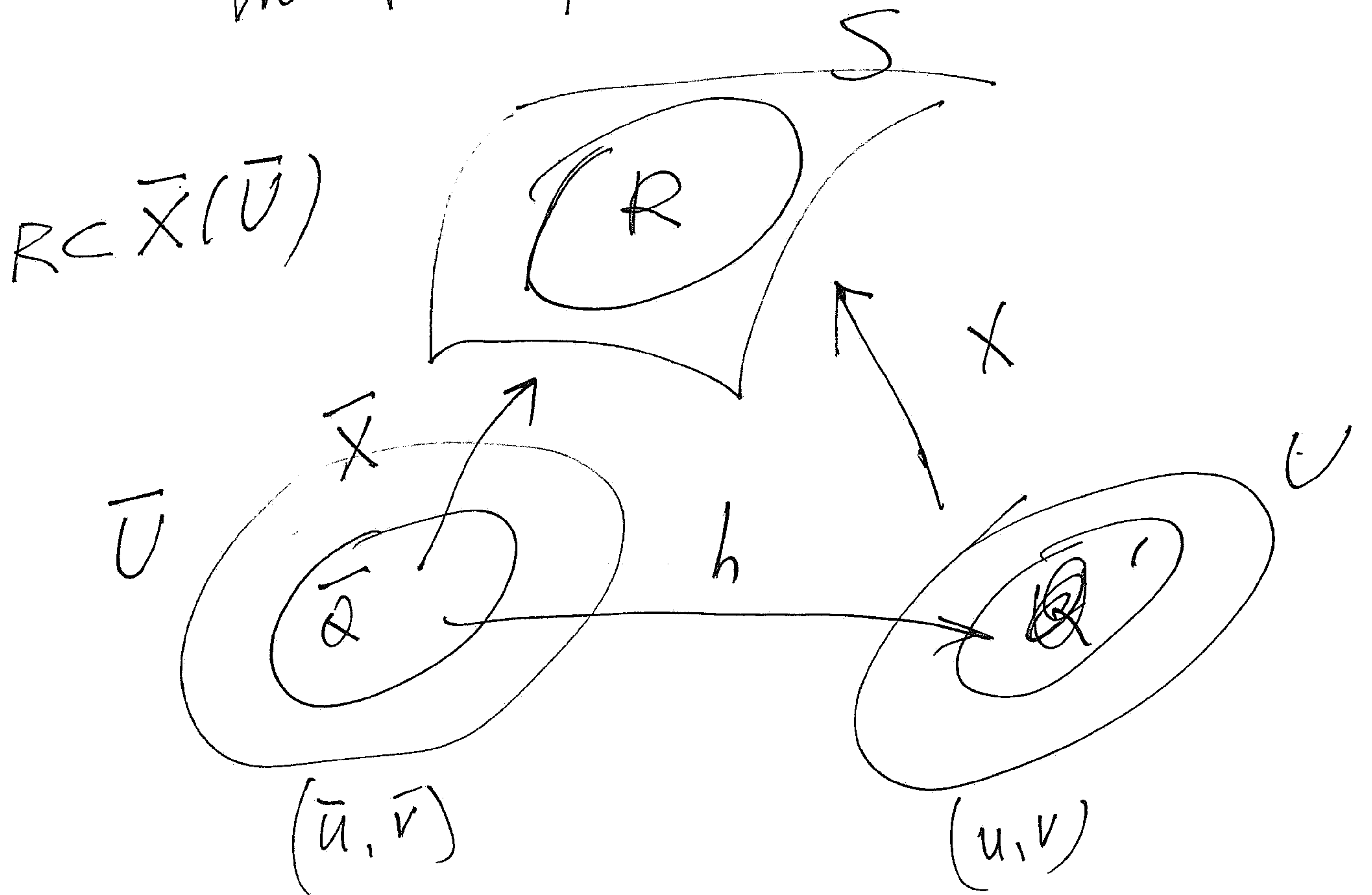
$Q \subset U \qquad R = X(Q) \subset S$

$$\iint_Q |X_u \times X_v| \, du\, dv$$

indept of parametrization



Set $\bar{Q} = \bar{X}^{-1}(R) \qquad h = X^{-1} \circ \bar{X}$ diffeom.

with Jacobian $\dfrac{\partial(u, v)}{\partial(\bar{u}, \bar{v})}$

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$$\iint_{\bar{Q}} |\bar{X}_{\bar{u}} \times \bar{X}_{\bar{v}}| \, d\bar{u}\, d\bar{v} = \iint_{Q} |X_u \times X_v| \left|\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}\right| d\bar{u}\, d\bar{v}$$

change variable

$$= \iint_{Q} |X_u \times X_v| \, du\, dv$$

$$\bar{X} = X \circ h = X(u(\bar{u},\bar{v}), v(\bar{u},\bar{v}))$$

$$\bar{X}_{\bar{u}} = X_u \frac{\partial u}{\partial \bar{u}} + X_v \frac{\partial v}{\partial \bar{u}}$$

$$\bar{X}_{\bar{v}} = X_u \frac{\partial u}{\partial \bar{v}} + X_v \frac{\partial v}{\partial \bar{v}}$$

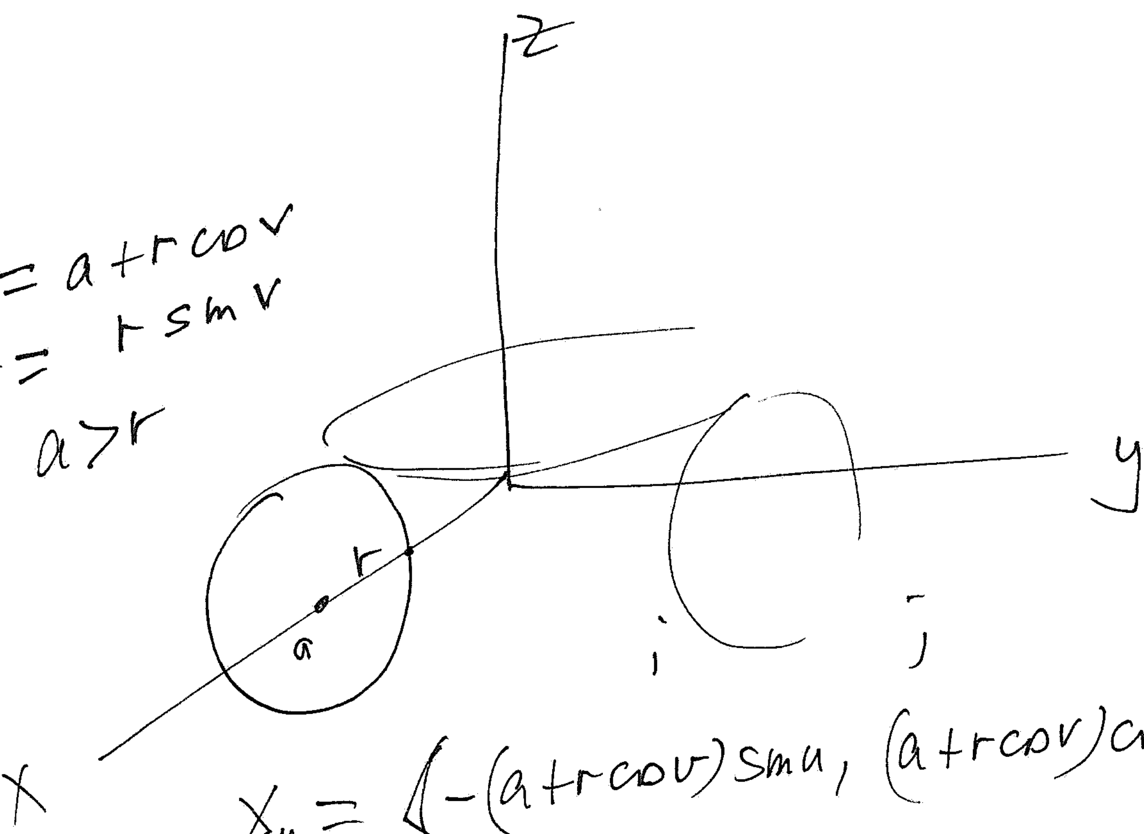
$$\bar{X}_{\bar{u}} \times \bar{X}_{\bar{v}} = (X_u \times X_v)\left(\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}\right)$$

Ex (torus of revolution)

$$X(u,v) = \big((a+r\cos v)\cos u,\ (a+r\cos v)\sin u,\ r\sin v\big)$$

$$0 < v < 2\pi$$
$$0 < u < 2\pi$$

$x = a + r\cos v$
$z = r\sin v$
$a > r$



$i \qquad j \qquad k$

$$X_u = \langle -(a+r\cos v)\sin u,\ (a+r\cos v)\cos u,\ 0\rangle$$
$$X_v = \langle -r\sin v\cos u,\ -r\sin v\sin u,\ r\cos v\rangle$$

$$X_u \times X_v = \langle r(a+r\cos v)\cos v\cos u,\ r(a+r\cos v)\cos v\sin u,\ r(a+r\cos v)\sin v$$

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$$|X_u \times X_v| = r(a + r\cos v)$$

$$A(\text{Torus}) = \int_0^{2\pi}\int_0^{2\pi} r(a + r\cos v)\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} (ar + r^2 \cos v)\, du\, dv$$

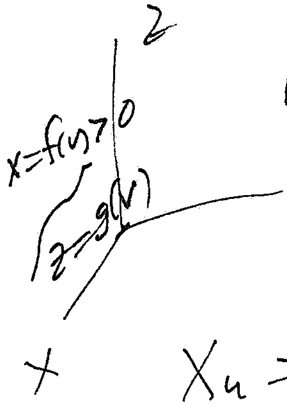
$$= 4\pi^2 r a$$

Remark $\quad |X_u \times X_v| = \sqrt{EG - F^2}$

$$E = X_u^2 \quad F = X_u \cdot X_v \quad G = X_v^2$$

surface area of graph in
general surface of revolution

$$X(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$$



$$|X_u \times X_v| = f(v)\sqrt{f'^2+g'^2}$$

$$a \le v \le b$$
$$0 \le u \le 2\pi$$

$$X_u = \langle -f(v)\sin u,\ f(v)\cos u,\ 0\rangle$$

$$X_v = \langle f'(v)\cos u,\ f'(v)\sin u,\ g'(v)\rangle$$

$$X_u \times X_v = \langle f(v)\,g'(v)\cos u,\ -f(v)g'(v)\sin u,\ -f(v)f'(v)$$

$$\text{Area} = \iint f(v)\sqrt{f'^2+g'^2}\,du\,dv$$

$$= 2\pi\int_a^b f(v)\sqrt{f'^2+g'^2}\,dv$$

$$E = f(v) \qquad F = 0 \qquad G = \sqrt{f'^2+g'^2}$$

Ex

$x = f(v) > 0$
$z = v$ $\quad a \le v \le b$

graph rotated around z-axis

$$\text{Surface Area} = 2\pi \int_a^b f(v)\sqrt{1+f'^2}\,dv$$

7.5 Integrals of scalars on a surface

Def'n $\iint_S f(x,y,z)\,dS = \iint_S f\,dS = \iint_D f(\vec{X}(u,v))\,\|\vec{X}_u \times \vec{X}_v\|\,du\,dv$

$= \iint_D f(\vec{X}(u,v))\sqrt{EG-F^2}\,du\,dv$

where f is continuous on nbhd U of S in $\mathbb{R}^3$ and $S = X(D)$ is a patch.

Ex 1 helicoid ~~$X(u,v) = (v\cos u, v\sin u, u)$~~ $0 \le u \le 2\pi$

$X(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$

$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$f(x,y,z) = \sqrt{1+x^2+y^2}$

Find $\displaystyle\iint_S f\,dS = \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\,\sqrt{1+r^2}\,dr\,d\theta$

$E = X_r^2 = 1$ $\qquad \sqrt{EG-F^2} = \sqrt{r^2+a^2}$

$F = X_r \cdot X_\theta = 0$

$G = X_\theta^2 = r^2 + a^2 \qquad f(r,\theta) = \sqrt{1+r^2}$

$$= 2\pi \int_0^1 (1+r^2)\,dr = 2\pi \cdot \tfrac{4}{3} = \tfrac{8\pi}{3}$$

Ex 2 $\quad$ Let S: $\quad z = x^2 + y$

D: $\quad 0 \le x \le 1$

$-1 \le y \le 1$

Evaluate $\displaystyle\iint_D x\,dS = \iint_D x\sqrt{2+4x^2}\,dx\,dy$

$$= 2\int_0^1 x\sqrt{2+4x^2}\,dx$$

$$= \frac{1}{4}\,\frac{(2+4x^2)^{3/2}}{3/2}\Bigg|_0^1 = \frac{1}{6}\left(6^{3/2} - 2^{3/2}\right)$$

$$= \sqrt{6} - \frac{\sqrt{2}}{3}$$

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Ex 3 Evaluate $\iint_S z^2 \, dS$

S = unit sphere $x^2+y^2+z^2=1$

We use spherical coordinates $\rho = 1$

$$X(\theta,\phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$

$$\|X_\theta \times X_\phi\| = \sin\phi$$

$$\iint_S z^2 \, dS = \int_0^{2\pi}\int_0^{\pi} \cos^2\phi \sin\phi \, d\phi \, d\theta$$

$$= 2\pi \int_0^{\pi} \cos^2\phi \sin\phi \, d\phi$$

$$= -2\pi \frac{\cos^3\phi}{3}\Big|_0^{\pi} = \frac{4}{3}\pi$$

Remark If $S =$ sphere of radius R

$$\iint_S f\,dS = \int_0^{2\pi}\int_0^{\pi} f(\phi,\theta)\,R^2 \sin\phi\, d\phi\, d\theta$$

Integrals over graphs

Let $S: z = g(x,y)$ $(x,y) \in D$

and let $f(x,y,z)$ be given.

Then $\displaystyle\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y)) \cdot \sqrt{1+|\nabla g|^2}\,dx\,dy$

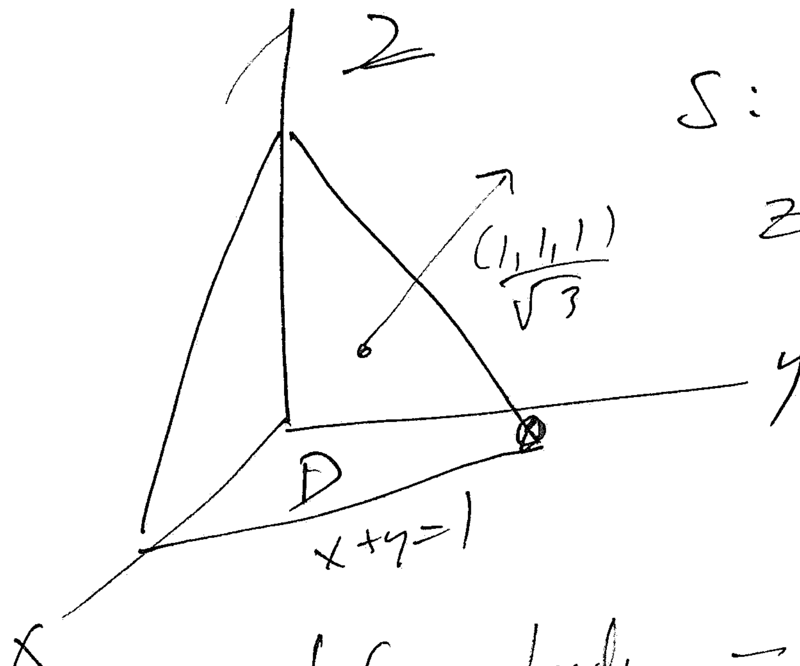
$$= \iint_D \frac{f(x,y,g(x,y))}{\cos\theta}\,dx\,dy$$

where $\cos\theta = \underbrace{\vec{N}}_{\substack{\text{upward unit} \\ |\vec{N}|=1}} \cdot \vec{k} = \dfrac{1}{\sqrt{1+|\nabla g|^2}}$

so $\sqrt{1+|\nabla g|^2} = \dfrac{1}{\cos\theta}$

Ex 4 Compute $\iint_S x \, dS$

where S is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$



$S: \ x+y+z=1$

$z = 1-(x+y)$

$\cos\theta = \vec{N}\cdot\vec{n}$

$= \frac{1}{\sqrt{3}}$

$$= \sqrt{3}\iint_D x \, dx\,dy = \sqrt{3}\int_0^1\int_0^{1-x} x \, dy\,dx$$

$$= \sqrt{3}\int_0^1 x(1-x)\,dx$$

$$= \sqrt{3}\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{\sqrt{3}}{6}$$

Ex 5 Let S be the helicoid of Ex 2.

$$X(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$$

$$0 \le \theta \le 2\pi$$

$$0 \le r \le 1$$

Suppose S is made of a material of mass density at (x,y,z)

$$= 2\sqrt{x^2+y^2}$$

~~Use cylindrical coord~~ to find the total mass

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$$M = \iint_S 2\sqrt{x^2+y^2}\, dS$$

$$= \iint_D 2r\sqrt{1+r^2}\, dr\, d\theta$$

$$= 2\pi \int_0^1 2r\sqrt{1+r^2}\, dr$$

$$= 2\pi \tfrac{2}{3}(1+r^2)^{3/2}\Big|_0^1$$

$$= 4/3\,\pi\left(2^{3/2}-1\right).$$

7.6

Surface integrals of vector fields

Def'n Let $\vec{F}$ be a vector field defined on $S = \vec{X}(D)$, $D \subset \mathbb{R}^2$. Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{X}_u \times \vec{X}_v \, du\, dv$$

$$= \iint_S \vec{F} \cdot \vec{N} \, dS \,,$$

where $\vec{N}$ is the unit normal induced by the parametrization $\vec{X}$ (i.e. $\vec{N} = \dfrac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|}$)

Ex 1 Let D be the rectangle

$$0 \leq \theta \leq 2\pi \,, \quad 0 \leq \phi \leq \pi$$

(spherical parametrization of the unit sphere)

and $\vec{X} : D \to \mathbb{R}^3$ given by

$$\vec{X}(\theta, \phi) = (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi)$$

Compute $\iint_S \vec{X} \cdot d\vec{S}$

$$\vec{X}_\theta = (-\sin\phi \sin\theta \,, \sin\phi \cos\theta \,, 0)$$

$$\vec{X}_\phi = (\cos\phi \cos\theta, \cos\phi \sin\theta, -\sin\phi)$$

$$\vec{X}_\theta \times \vec{X}_\phi = \langle -\sin^2\phi \cos\theta, -\sin^2\phi \sin\theta, -\sin\phi \cos\phi \rangle$$

so $\vec{N} = -\vec{X}$ = inner unit normal

$$|\vec{X}_\theta \times \vec{X}_\phi| = \sin\phi$$

$$\iint_S \vec{X}\cdot d\vec{S} = \iint_S \vec{X}\cdot\vec{N}\, dS$$
$$= -\iint_S dS = -A(S) = -4\pi$$

Defn (local orientation) We may specify one of the two possible unit normals $\pm \dfrac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|}$ as being the positive (or preferred) orientation and the other as negative orientation

The Mobius band is an example of a surface with boundary that is not globally orientable

Th'm Every compact surface (without bdry) in $\mathbb{R}^3$ is globally orientable

For example if we choose $\vec{N} = \vec{X}$ in Ex. 1 pointing to the "outside" of the unit sphere, then we saw that $\dfrac{\vec{X}_\theta \times \vec{X}_\phi}{|\vec{X}_\theta \times \vec{X}_\phi|} = -\vec{N}$

(. orientation reversing)

The outer orientation is often the standard orientation for the sphere of radius R

<u>Ex 3</u> $S: \quad z = g(x,y)$

$$\vec{N} = \frac{\langle -\nabla g, 1 \rangle}{\sqrt{1 + |\nabla g|^2}}$$ is

the "upward unit normal"

to the graph and is

usually the preferred

orientation.

Thm (Independence of Surface integrals on parametrization)

Let S be an oriented surface and let $\vec{X}_1, \vec{X}_2$ be two regular orientation preserving parametriz. If $\vec{F}$ is a continuous vector field on S, then

$$\iint_{S_1 = X_1(D_1)} \vec{F} \cdot d\vec{S} = \iint_{S_2 = \vec{X}_2(D_2)} \vec{F} \cdot d\vec{S}$$

If $\vec{X}_1, \vec{X}_2$ have opposite orientations then

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = - \iint_{S_2} \vec{F} \cdot d\vec{S}$$

However if f a continuous real valued function, then

$$\iint_{S_1} f \, dS = \iint_{S_2} f \, dS$$

__Thm__ $\iint\limits_S \vec{F}\cdot d\vec{S} =$

$$\iint\limits_S \vec{F}\cdot\vec{N}\, dS$$

__Remark__ $\iint\limits_S \vec{F}\cdot \vec{dS}$ is the net flux across the surface (per unit time if $\vec{F}$ is thought of as a velocity) in the given orientation

Ex 4 Let $T(x,y,z) = x^2+y^2+z^2$

be a temperature function and

let S be unit sphere with

the outward orientation.

Find the heat flux

of $\vec{F} = -k \nabla T$ across S

(say $k=1$)

$\vec{F} = -\langle 2x, 2y, 2z \rangle$

$\vec{N} = \langle x, y, z \rangle$ on S

so $\vec{F} \cdot \vec{N} = -2(x^2+y^2+z^2) = -2$

on S so

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{N} \, dS$$

$$= -2 \iint_S dS = -8\pi \;//.$$

Suppose $\quad S = \text{graph}(g)$

$x, y \in D$

$$N = \frac{\langle -Dg, 1 \rangle}{\sqrt{1 + |Dg|^2}}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{N} \, dS$$

$$= \iint_D \left(\frac{-F_1 g_x - F_2 g_y + F_3}{\sqrt{1+|Dg|^2}} \right) \sqrt{1+|Dg|^2} \, dx\,dy$$

$$= \iint_D (F_3 - F_1 g_x - F_2 g_y) \, dx\,dy$$

Ex 6 Let S be the disk of radius 5 in the plane $z=12$

Let $\vec{r}=\langle x,y,z\rangle$ Compute

$$\iint_S \vec{r}\cdot d\vec{S} = \iint_D 12\,dx\,dy = 12\cdot\pi\cdot 25 = 300\pi$$

$D:\ x^2+y^2 \le 25$

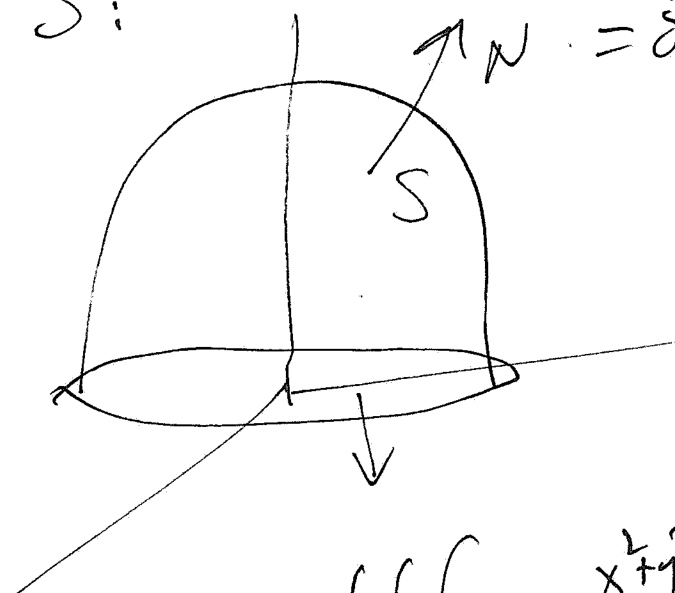
Ex

$$\vec{F}(x,y,z) = \langle xz,\ yz,\ 1+e^{x^2+y^2} \rangle$$

$$S:\quad z = 1-x^2-y^2\ ,\quad z \geq 0$$

N $= g(x,y)$

oriented upward



$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{x^2+y^2\le 1} \left(1+e^{x^2+y^2} - x(1-x^2-y^2)\cdot(-2x) - y(1-x^2-y^2)(-2y) \right) dx\,dy$$

$$= \iint_{x^2+y^2\le 1} \left(1+e^{x^2+y^2} + 2(1-x^2-y^2)(x^2+y^2) \right) dA$$

$2(x^2+y^2) - 2(x^2+y^2)^2$

$$= \int_0^{2\pi}\int_0^1 \left(1+e^{r^2}+2r^2(1-r^2)\right) r\,dr\,d\theta$$

$$= 2\pi \left(\frac{r^2}{2}+\frac{1}{2}e^{r^2}+\frac{r^4}{2}-\frac{r^6}{3}\right)\Big|_0^1$$

$$= 2\pi\left(\tfrac{1}{2}(e-1)+1-\tfrac{1}{3}\right) = 2\pi\left(\tfrac{e}{2}+\tfrac{1}{6}\right)$$

$$= \pi\left(e+\tfrac{1}{3}\right).$$

divergence theorem

$$\iiint_W \operatorname{div}\vec{F}\,dV = \iint_{\partial W} \vec{F}\cdot\vec{N}\,dS$$

outer normal

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$$\operatorname{div} \vec{F} = z + z + 0 = 2z$$

$$\iiint_W 2z \, dS = \iint_{S + \text{disk}} \vec{F} \cdot \vec{N} \, dS$$

$$= \iint_{x^2+y^2 \le 1} 2\left(1 - (x^2+y^2)\right) dA - \iint_{x^2+y^2 \le 1} \left(1 + e^{x^2+y^2}\right)$$

$$= 2 \cdot 2\pi \int_0^1 (1 - r^2) \, r \, dr$$

$$= 4\pi \cdot \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1$$

divergence theram check

$$\iiint_W 2z\, dV = \iint_S \vec{F}\cdot\vec{N}\, dS - \iint_{x^2+y^2\le 1} \left(1+e^{x^2+y^2}\right) dxdy$$

$$\parallel$$

$$\iint_{x^2+y^2\le 1}\left(\int_0^{(1-x^2-y^2)} 2z\, dz\right) dxdy = \iint_{x^2+y^2\le 1} \left(1-\cancel{x}^2-y^2\right)^2 dxdy$$

$$\iint_S \vec{F}\cdot\vec{N}\, ds = \iint_{x^2+y^2\le 1}\left(\left(1+e^{x^2+y^2}\right) + \left(1-2(x^2+y^2)+(x^2+y^2)^2\right)\right) dA$$

$$= 2\pi\left\{\frac{1}{2}(e-\cancel{1}) + \cancel{\frac{1}{2}}+\frac{1}{6}\right\} = \pi\left(e+\frac{1}{3}\right) ✓$$

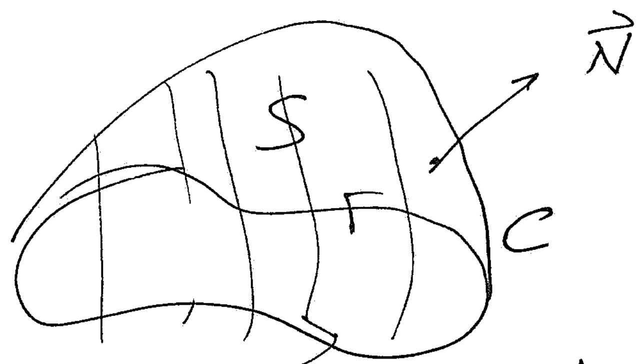
divergence theorem in $\mathbb{R}^3$

$$\iiint_W \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}$$

$$= \iint_S \vec{F} \cdot \vec{N}\, dS$$

outer unit normal to S.

Stokes thm



$$\int_C \vec{F} \cdot d\vec{S} = \iint_S (\operatorname{curl} \vec{F}) \cdot \vec{N}\, dS$$

$$\| \qquad\qquad \|$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

Ex 1 $\vec{F}(x,y,z) = \langle y, x, z \rangle$

S_1: $z = 1 - x^2 - y^2$

$S = \partial W$ oriented out

Compute $\iint_S \vec{F} \cdot d\vec{S}$

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{D:\, x^2+y^2 \le 1} \left(\cancel{z} - y(-2x) - x(-2y) + 1 - x^2 - y^2 \right) dx\,dy$$

$$= \iint_D \left(-4xy + 1 - x^2 - y^2 \right) dA$$

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = -\iint_D z \, dA = 0$$

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$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \Big(-4r^2 \underset{0}{\cancel{\sin\theta\cos\theta}} + (1-r^2)\Big)\, r\,dr\,d\theta$$

$$= 2\pi\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1 = \pi/2$$

check (divergence thm)

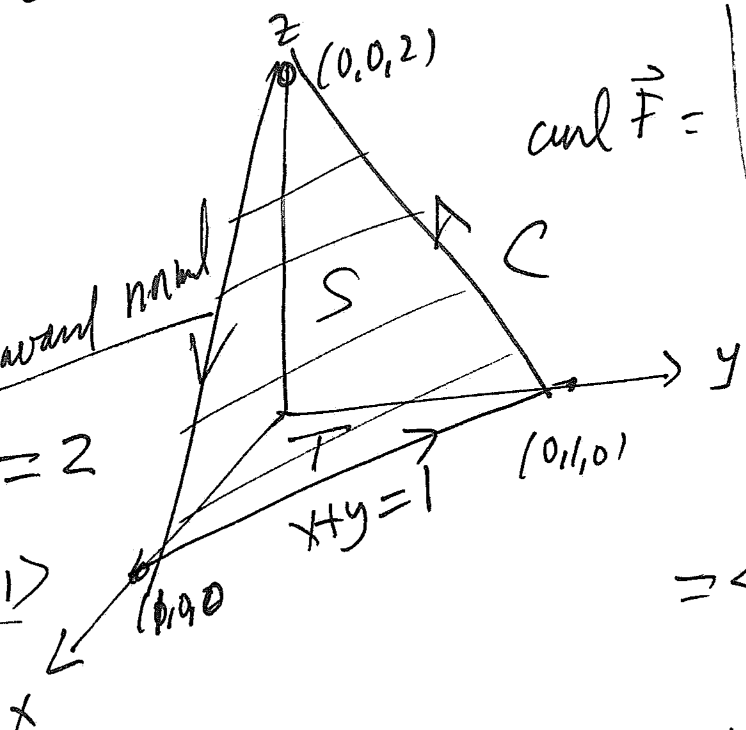
$$\operatorname{div}\vec{F} = 1 \qquad \iiint_W \operatorname{div}\vec{F}\,dV = \operatorname{Vol} W$$

cylind.

$$= \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta = 2\pi\int_0^1 r(1-r^2)\,dr = \pi/4$$

Ex Use Stokes to evaluate

$$\iint_S \text{curl}\,\vec{F}\cdot\vec{N}\,dS \quad , \quad \vec{F} = \langle z^2, y^2, xy \rangle$$



$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ z^2 & y^2 & xy \end{vmatrix}$$

$$= \langle x, 2z-y, 0 \rangle$$

plane upward normal

$2x+2y+z=2$

$$\vec{N} = \frac{\langle 2,2,1 \rangle}{3}$$

$$\text{curl}\,\vec{F}\cdot N = \frac{2x+2(2z-y)}{3}$$

$$z = 2(1-x-y) = g \qquad dS = \sqrt{1+|\nabla g|^2}\,dA = 3\,dA$$

$$\iint_S \text{curl}\,\vec{F}\cdot N\,dS = \iint_T (2x+4z-2y)\,dA$$

$$= \iint_T (2x+4\cdot 2(1-x-y)-2y)\,dA$$

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$$= \iint_T (-6x - 10y + 8)\, dA$$

$$= \int_0^1 \int_0^{1-x} (8 - 6x - 10y)\, dy\, dx$$

$$= \int_0^1 \left[(8-6x)(1-x) - 5(1-x)^2 \right] dx$$

$$= \int_0^1 (1-x)(8 - 6x - 5 + 5x)\, dx$$

$$= \int_0^1 (1-x)(3-x)\, dx$$

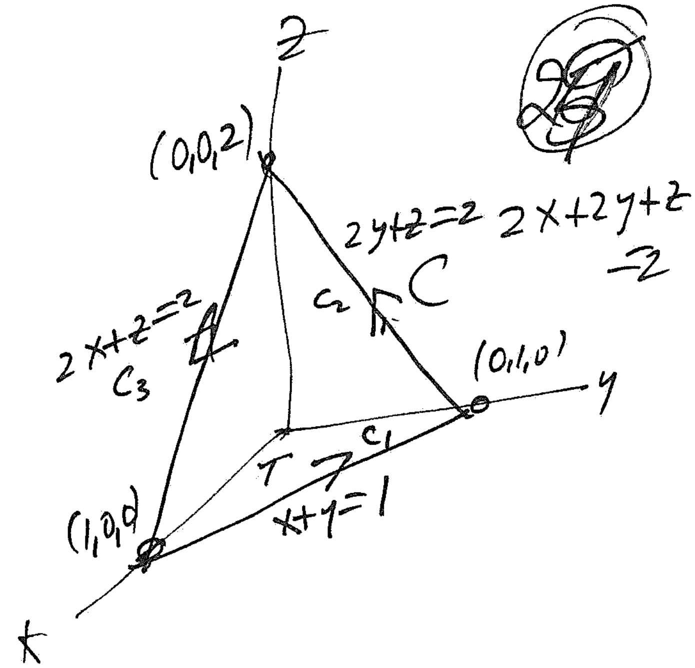
$$= \int_0^1 (3 - 4x + x^2)\, dx$$

$$= \left(\frac{x^3}{3} - 2x^2 + 3x \right) \Big|_0^1 = \frac{4}{3}$$

$$\vec{F} = \langle z^2, y^2, xy \rangle$$

$$\int \vec{F} \cdot d\vec{s}$$

$$C = c_1 + c_2 + c_3$$



$$= \int_C z^2 dx + y^2 dy + xy \, dz$$

$$= \underbrace{\int_0^1 y^2 dy}_{c_1} \cancel{+} \underbrace{\int_0^1 y^2 dy}_{c_2} + \underbrace{\int_0^2 \frac{1}{2} z^2 dz}_{c_3}$$

$$= \frac{1}{2} \cdot \frac{z^3}{3} \Big|_0^2 = 4/3$$