7.1 The path integral

Def'n Let $f : \mathbb{R}^3 \to \mathbb{R}$
and let $\vec{c} : I = [a, b] \to \mathbb{R}^3$
be a C^1 path in $\mathbb{R}^3$

The path integral of f over $\vec{c}$,

$$\int_{\vec{c}} f\, ds = \int_a^b f(\vec{c}(t))\,|\vec{c}\,'(t)|\,dt$$

Remark If $\vec{c}(t)$ is only piecewise C^1, i.e $\exists$ partition

$a = t_0 < t_1 < \dots < t_N = b$ ②

such that $\vec{c}\,|_{[t_{i-1}, t_i)}$ is C^1

$i = 1, \dots, N$ and $\vec{c}$ is continuous on $[a,b]$

When $f \equiv 1$ we recover the definition of the arc length of $\vec{c}$

Ex 1 Let $\vec{c} : [0, 2\pi] \to \mathbb{R}^3$ be the helix $\vec{c}(t) = (\cos t, \sin t, t)$ and $f(x,y,z) = x^2+y^2+z^2$. Then

$f(\vec{c}(t)) = 1 + t^2$, $|\vec{c}\,'(t)| = \sqrt{2}$

so $\int_{\vec{C}} f\,ds = \int_0^{2\pi} (1+t^2)\sqrt{2}\,dt$

$= \sqrt{2}\left(t+\frac{t^3}{3}\right)\Big|_0^{2\pi} = 2\pi\sqrt{2}\left(1+\frac{4\pi^2}{3}\right)$ //

Without loss of generality, assume $\vec{c}$ is of class $\mathcal{C}^1$ on I. Then form the Riemann sums (based on the inscribed polygons as we did for arc length):

$$S_N = \sum_{i=0}^{N} f(x_i, y_i, z_i)\,\Delta s_i$$

where $\Delta s_i = \int_{t_i}^{t_{i+1}} |c'(t)|\,dt$ is the arc length of $c(t_i, t_{i+1})$ $\quad 0 \le i \le N-1$

and $(x_i, y_i, z_i) = c(t)$ for some $t \in [t_i, t_{i+1}]$

$\Delta t_i = t_{i+1} - t_i$

By the MVT,

$$\Delta s_i = |c'(t_i^*)| \, \Delta t_i \quad \text{for some } t_i^* \in [t_i, t_{i+1}]$$

Then for f continuous, we know

the Riemann sums converge, i.e.

$$\lim_{N \to \infty} S_N = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(x_i, y_i, z_i) \, |c'(t_i^*)| \, \Delta t_i$$

$$= \int_a^b f(x(t), y(t), z(t)) \, |c'(t)| \, dt$$

$$= \int_{\vec{c}} f \, ds \quad //$$

The case of planar curves

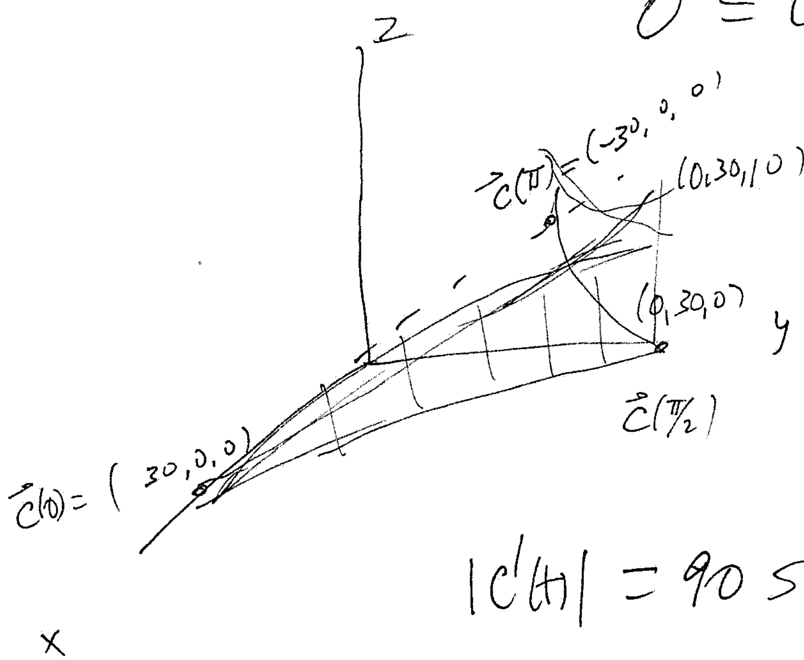
Suppose that $\vec{c}(t) = (x(t), y(t))$ lies in the x,y plane and $f : \mathbb{R}^2 \to \mathbb{R}$

Then $$\int_{\vec{c}} f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{x'^2(t) + y'^2(t)}\, dt$$

When $f \geq 0$ we can interpret $\int_{\vec{c}} f\, ds$ as the (one-sided) area of a "fence" of variable height $f(x,y)$ and base $\vec{c}$

Ex 2 Let $\vec{c}(t) = (30\cos^3 t,\ 30\sin^3 t)$

$0 \le t \le \pi/2$



$f(x,y) = 1 + \frac{y}{3}$

$\vec{c}'(t) = (-90\cos^2 t \sin t,\ 90\sin^2 t \cos t)$

$$|\vec{c}'(t)| = 90 \sin t \cos t$$

The area of one side of the fence in the first quadrant is then

$$\int_C \left(1+\frac{y}{3}\right) ds = \int_0^{\pi/2} \left(1 + \frac{30\sin^3 t}{3}\right) 90 \sin t \cos t \, dt$$

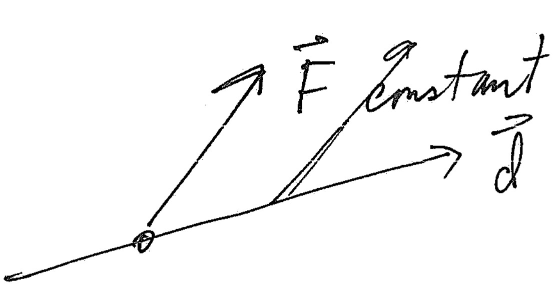
$$= 90\int_0^{\pi/2} (\sin t + 10\sin^4 t)\cos t \, dt$$

$$= 90\left(\frac{\sin^2 t}{2} + 2\sin^5 t\right)\Big|_0^{\pi/2} = 90\left(\frac{1}{2}+2\right) = 225$$

7.2 Line Integrals

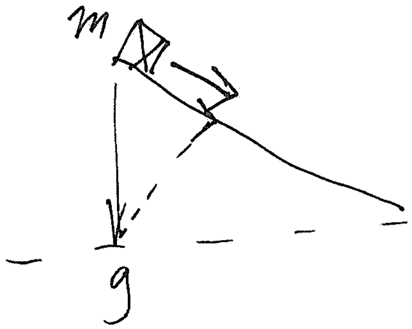
We next discuss integration of vector fields along a path and in particular the work done by a force in moving a particle along a path.

For a straight path, in a constant force field $\vec{F}$, the



work done by the force in moving

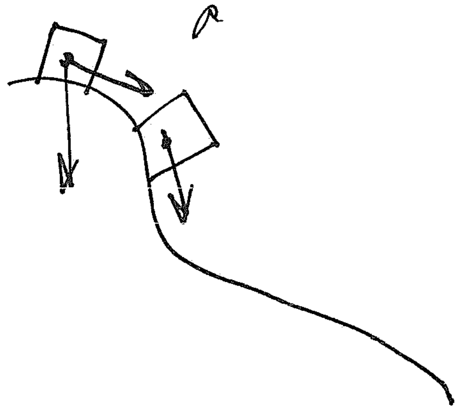
(displacing) a particle is $\vec{F} \cdot \vec{d}$ ("rule of Force times distance")

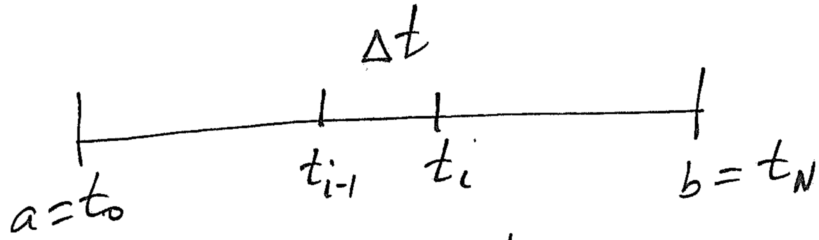


In the case of a curved path, we approximate the path by (infinitesimal) straight paths (inscribed polygons) and take the tangential component of $\vec{F}$. That is for $\vec{c} : [a,b] \to \mathbb{R}^3$

$$W \;(= \text{work done by } \vec{F}) = \int_{\vec{c}} \vec{F} \cdot d\vec{s}$$

$$= \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$





$$\Delta \vec{s} \approx \vec{c}\,'(t)\,\Delta t$$

$$\vec{F}(\vec{c}(t)) \cdot \Delta \vec{s} = \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,\Delta t$$

$$W \approx \sum_{i=0}^{N-1} \vec{F}(\vec{c}(t_i)) \cdot \Delta \vec{s} = \sum_{i=0}^{N-1} \vec{F}(\vec{c}(t_i)) \cdot \vec{c}\,'(t_i)\,\Delta t$$

$$\xrightarrow{N \to \infty} \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt$$

<u>Def'n</u> Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ and $\vec{c} : [a,b] \to \mathbb{R}^3$ a (piecewise) C^1 path. (oriented)

Then the line integral of $\vec{F}$ along $\vec{c}$ is $\displaystyle\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt$

If $\vec{c}\,'(t) \neq 0$, then

we may write $T(t) = \dfrac{\vec{c}\,'(t)}{|\vec{c}\,'(t)|} =$

unit tangent vector of $\vec{c}$ and

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

$$= \int_a^b \underbrace{\vec{F}(\vec{c}(t)) \cdot T(t)}_{\substack{\text{tangential} \\ \text{component of } \vec{F} \\ \text{along } \vec{c}}} \underbrace{\|\vec{c}\,'(t)\|\, dt}_{ds}$$

<u>Ex 1</u> $\vec{c}(t) = (\sin t, \cos t, t)$

$0 \leq t \leq 2\pi$

$\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$

= "radial v.f."

Then $\vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) =$

$\langle \sin t, \cos t, t \rangle \cdot \langle \cos t, -\sin t, 1 \rangle$

$= t$, so

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} t\, dt = 2\pi^2$$

Sometimes one writes

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} \left(F_1\, dx + F_2\, dy + F_3\, dz \right)$$

"differential form"

$$= \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$$

<u>Ex 2</u>

$$\int_{\vec{c}} x^2\, dx + xy\, dy + dz$$

$$\vec{c}(t) = (t, t^2, 1) \qquad 0 \le t \le 1$$

$$= \int_0^1 \left(t^2 \cdot 1 + t^3 \cdot 2t + 1 \cdot 0 \right) dt$$

$$= \int_0^1 \left(t^2 + 2t^4 \right) dt = \frac{11}{15}$$

Ex 3

$$\int_{\vec{C}} \cos z\,dx + e^x dy + e^y dz$$

$$\vec{C}(t) = (1, t, e^t), \quad 0 \le t \le 2$$

$$= \int_0^2 \left(\cos e^t \cdot 0 + e \cdot 1 + e^t \cdot e^t\right) dt$$

$$= \int_0^2 (e + e^{2t})\,dt = 2e + \frac{1}{2}(e^4 - 1)$$

Ex 4

$\vec{C}$: $x = \cos^3\theta \quad y = \sin^3\theta \quad z = \theta$

$$0 \le \theta \le \tfrac{7}{2}\pi$$

$$\int_C \left(\sin z\,dx + \cos z\,dy - (xy)^{1/3} dz\right)$$

$$= \int_0^{7/2\pi} \left(\sin\theta \cdot (\cancel{-3\cos^2\theta} \sin\theta) + \cos\theta \cdot \cancel{3\sin^2\theta}\cos\theta - \sin\theta\cos\theta\right) d\theta$$

$$= -\frac{1}{2}\int_0^{7/2\pi} \sin 2\theta\, d\theta$$

$$= \frac{1}{4}\cos 2\theta \Big|_0^{7/2\pi} = -\frac{1}{2}$$

Remark $W < 0$ means the force impedes the movement.

Reparametrization

Recall that a reparametrization of $\vec{c}$ is a 1-1 onto mapping h of $I' = [a', b']$ onto $I^\circ = [a^\circ, b^\circ]$

$$\xrightarrow{t^\circ = h(t')}$$

a' t' b a° t b°

~~$t = t(t)$ inverse mapping~~

~~The curve~~ If $a' \mapsto a^\circ$, $b' \mapsto b^\circ$

we say the

reparametrization is orientation preserving; if $a' \mapsto b$, $b' \mapsto a$

we say it is orientation reversing

The composition $\vec{p} = \vec{c} \circ h : I \to R^3$

e.g. $s(t) = \int_a^t |c'(\tau)|\, d\tau$ is the arc length function $0 \leq s \leq L$

$\tilde{c}(s) = c \circ t(s)$

Theorem (repar. of a line integral)

If h is orientation preserving

$$\int_{\vec{p}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} \vec{F} \cdot d\vec{s}$$

while if h is orientation

reversing, then

$$\int_{\vec{p}} \vec{F} \cdot d\vec{s} = -\int_{\vec{c}} \vec{F} \cdot d\vec{s}$$

Pf

$$\int_{\vec{p}} \vec{F} \cdot d\vec{s} = \int_{a'}^{b'} \vec{F}(p(t)) \cdot \vec{p}'(t')\,dt'$$

$$\vec{p}(t') = \vec{c}(h(t'))$$

$$\frac{d}{dt'} \vec{p} = \vec{c}'(h(t')) \, h'(t')$$

$$= \int_{a'}^{b'} \vec{F}(\vec{c}(h(t'))) \, c'(h(t')) \, h'(t')\,dt'$$

Set $s = h(t')$

$$= \pm \int_a^b \vec{F}(\vec{c}(s)) \, \vec{c}'(s)\,ds$$

Remark The path integral

$$\int_{\vec{c}} f(x,y,z)\,ds \quad \text{is}$$

indep't of path, i.e. it does not depend on the orientation of $\vec{c}$.

Theorem ($\vec{F} = \nabla f$ is (conservative) indep't of path)

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

depends only on the endpts of $\vec{c}$

Pf $\int_a^b \vec{F}(c(t)) \cdot \vec{c}'(t)\, dt$

$= \int_a^b \nabla f(c(t)) \cdot \vec{c}'(t)\, dt$

chain rule $= \int_a^b \left(\frac{d}{dt} f(\vec{c}(t)) \right) dt$

$= f(\vec{c}(b)) - f(\vec{c}(a))$ //

Ex $\vec{F} = (y, x, 0)$

$= \nabla(xy)$

so if $\vec{c}(t) = \left(t^4/4,\ \sin^3 \frac{t\pi}{2},\ e^{e^t} \right)$

$0 \le t \le 1$

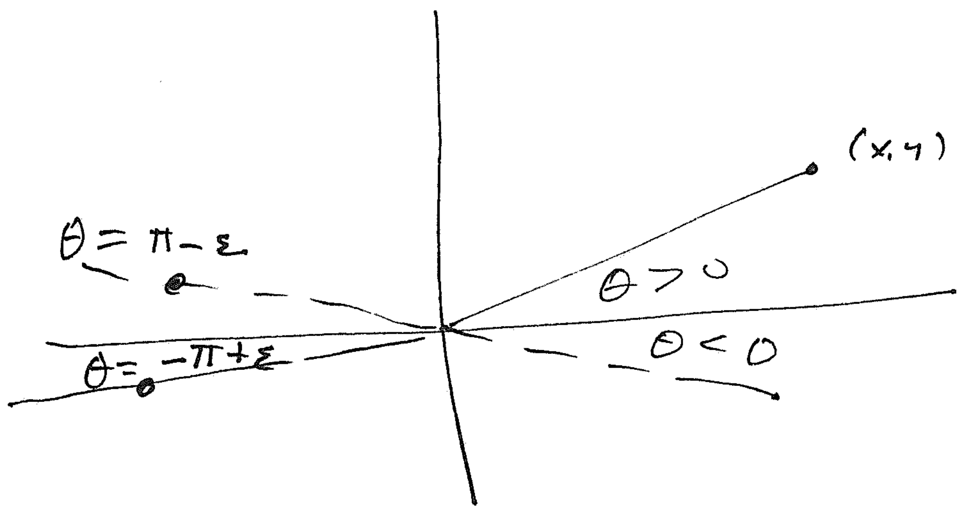
$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = f(\vec{c}(1)) - f(\vec{c}(0))$

$= \frac{1}{4} \cdot 1 - 0 = 1/4$

(sophisticated)

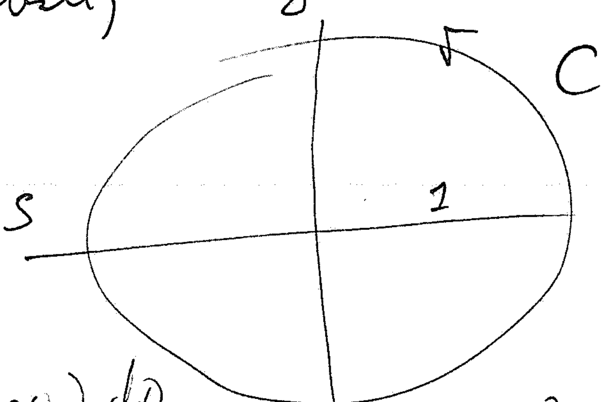
<u>Example</u> $\vec{F} = \frac{-y\,i}{x^2+y^2} + \frac{x}{x^2+y^2}\,j$

$= \nabla\theta \qquad \theta = \tan^{-1}\frac{y}{x}$



θ is not a globally single-valued <u>continuous function</u>

$$\int_C \vec{F}\cdot ds$$

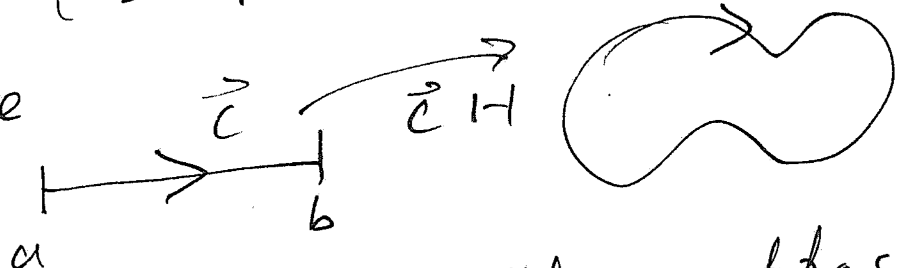


$x = \cos\theta$
$y = \sin\theta$

$$= \int_0^{2\pi} (-\sin\theta \cdot -\sin\theta + \cos\theta\cdot\cos\theta)\,d\theta = \int_0^{2\pi} 1\,d\theta = 2\pi \;!$$

line integrals over oriented closed curves
(simple or not)

Assume $\vec{c}$ is piecewise C^1

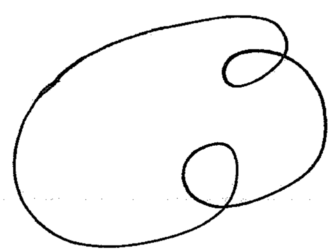


$c(\vec{a}) = c(\vec{b})$

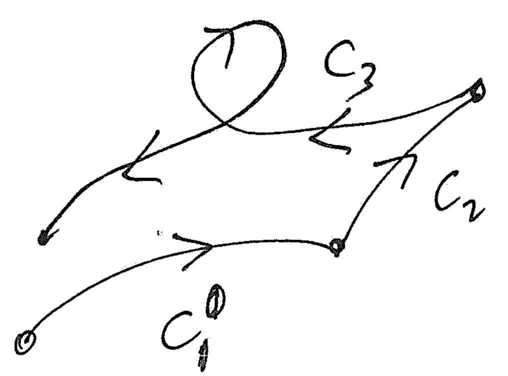
2 possible orientations

(smooth if $\vec{c}'(a) = \vec{c}'(b)$)

closed curve (not nec. 1-1)



$$\int_C \vec{F}\cdot d\vec{s} = \int_{C_1} \vec{F}\cdot d\vec{s} + \int_{C_2} \vec{F}\cdot d\vec{s} + \int_{C_3} \vec{F}\cdot d\vec{s}$$



$C = C_1 + C_2 + C_3$

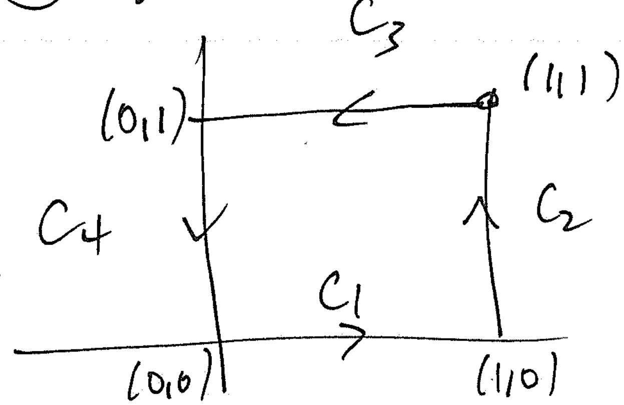
other notation:

$$\int_C \vec{F}\cdot d\vec{r} \qquad \vec{r} = \text{position vector}$$
$$= x\vec{i} + y\vec{j} + z\vec{k}$$

C : $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(t)\cdot\frac{d\vec{r}}{dt}\,dt$$

Ex :



$$\int_C x^2dx + xy\,dy =$$

method 1

$$\vec{c} : [0,4] \to \mathbb{R}^2$$

$$t \mapsto \begin{cases} (t,0) & 0 \le t \le 1 \\ (1, t-1) & 1 \le t \le 2 \\ (3-t, 1) & 2 \le t \le 3 \\ (0, 4-t) & 3 \le t \le 4 \end{cases}$$

$$\int_C x^2 dx + xy\, dy = \int_0^1 (t^2 + 0)\, dt + \int_1^2 (0 + t-1)\, dt + \int_2^3 (-(3-t)^2 + 0)\, dt + \int_3^4 (0+0)\, dt$$

method 2

$$\int_C x^2 dx + xy\, dy$$

$$= \cancel{\int_0^1 x^2 dx} + \int_0^1 1 \cdot y\, dy \quad \cancel{\int_0^1 x^2 dx}$$

$$\cancel{- \int_0^1 0 \cdot y\, dy}$$

$$= \frac{1}{2}$$

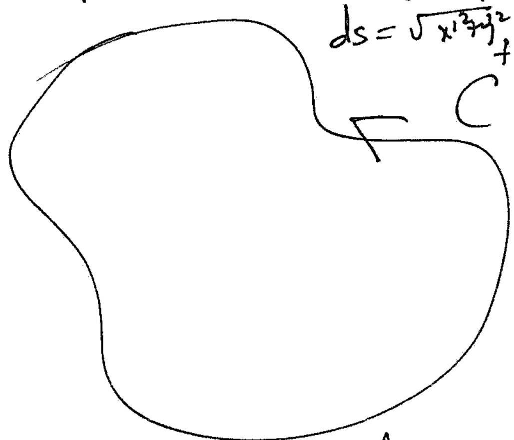
8.1 Green's theorem in the plane

Theorem 1 (Greens theorem version 1)

Let D be a simple region in the plane with positively oriented piecewise C^1 boundary C. Let P, Q be C^1 functions. Then

$$\int_{C^+} \underbrace{P\,dx + Q\,dy} = \iint_D (Q_x - P_y)\,dx\,dy$$

$\vec{F} = \langle P, Q \rangle$ $\quad \vec{T} = \frac{\langle x', y' \rangle}{\sqrt{x'^2 + y'^2}}$ $\quad ds = \sqrt{x'^2+y'^2}$ $\qquad (Q_x - P_y) = \operatorname{curl} \vec{F} \cdot \vec{k}$

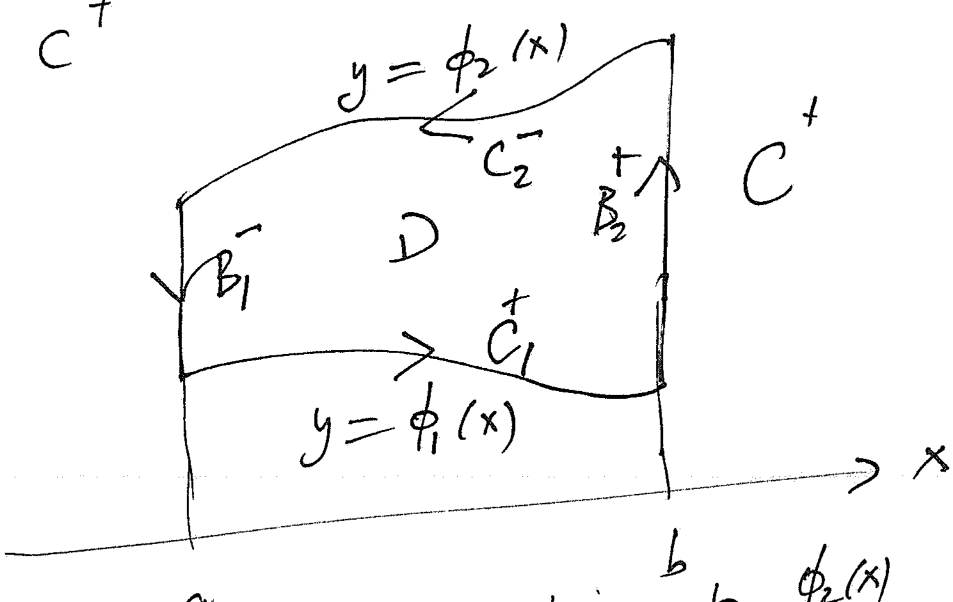


C^+

$$= \int_{C^+} \vec{F} \cdot d\vec{s} = \int_{C^+} \vec{F} \cdot \vec{T}\, ds = \int_{C^+} P\,dx + Q\,dy$$

<u>Lemma 1</u> Let D be y simple with boundary C. and $P(x,y) \in C^1$. Then

$$\int_{C^+} P\,dx = -\iint_D P_y\,dx\,dy$$



Then $\iint_D P_y\,dx\,dy \overset{\text{Fubini}}{=} \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} P_y(x,y)\,dy\,dx$

$$= \int_a^b \left[P(x, \phi_2(x)) - P(x, \phi_1(x)) \right] dx$$

Note $C_1^+ : \quad x \mapsto (x, \phi_1(x))$
is a graph $\qquad a \le x \le b$

so $\displaystyle\int_{C_1^+} P(x,y)\,dx = \int_a^b P(x, \phi_1(x))\,dx$

and similarly over top $\displaystyle\int_{C_2^+} P(x,y)\,dx = \int_a^b P(x, \phi_2(x))\,dx$

$$\parallel$$

$$-\int_{C_2^-} P(x,y)\,dx$$

Hence $\displaystyle\iint_D P_y\,dx\,dy = -\int_{C_1^+} P\,dx - \int_{C_2^-} P\,dx$

But $\displaystyle\int_{B_2^+} P\,dx = \int_{B_1^-} P\,dx = 0$ since x constant

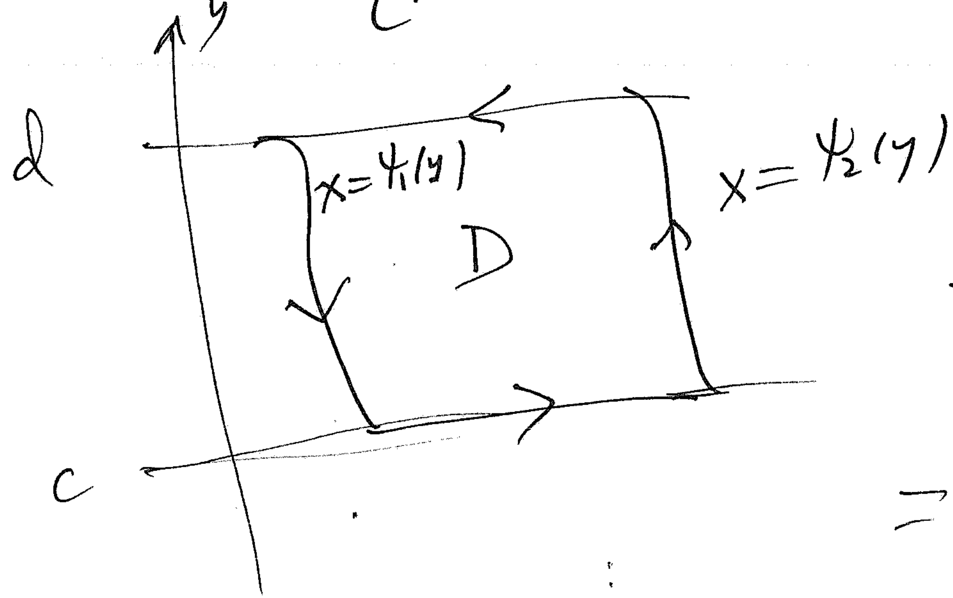
Hence

$$\int_{C^+} P\,dx = \int_{C_1^+} P\,dx + \int_{C_2^-} P\,dx = -\iint_D P_y\,dx\,dy \quad //.$$

<u>Similarly</u>

<u>Lemma 2</u> Let D be x-simple with positively oriented bdry C

Then $\displaystyle\int_{C^+} Q\,dy = \iint_D Q_x\,dx\,dy$



$$\iint_D Q_x\,dx\,dy = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} Q_x\,dx\,dy$$

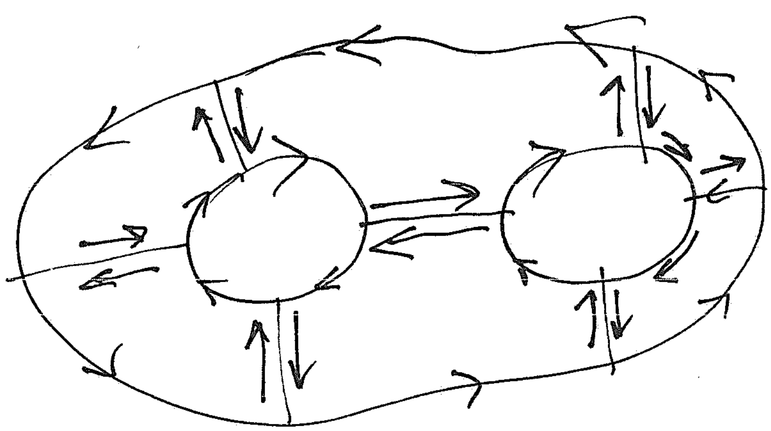
$$= \int_c^d \left(Q(\psi_2(y),y) - Q(\psi_1(y),y)\right) dy$$

$$= \int_{C^+} Q\,dy$$

Putting lemmas 1, 2 together yields

Thm 1.

Green's theorem easily extends to any region which can be decomposed into a finite # of regions which are simple



The flux (divergence) form of Green's theorem

Th 2 If $\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ is C^1 on D, then

$$\int_{C^+} \vec{F}\cdot\vec{N}\,ds = \iint_D \operatorname{div}\vec{F}\,dx\,dy = \iint_D (P_x + Q_y)\,dx\,dy$$

$\vec{N} =$ external unit normal to $C^+ = \partial D$

$P \to -Q$
$Q \to P$

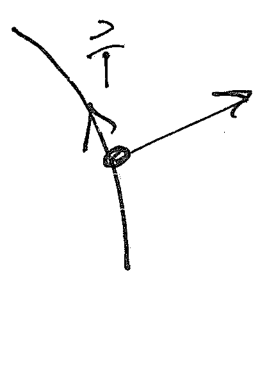
$\tilde{F} = \langle -Q, P\rangle$
$\vec{F} = \langle P, Q\rangle$

Pf By Th. 1

$$\iint_D (P_x + Q_y)\,dx\,dy = \int_{C^+} -Q\,dx + P\,dy$$

$$\vec{T} = \frac{\langle x', y'\rangle}{\sqrt{x'^2 + y'^2}}$$

$$\vec{N} = \frac{\langle +y', -x'\rangle}{\sqrt{x'^2 + y'^2}}$$



$$\tilde{F}\cdot\vec{T}\,ds = \langle -Q, P\rangle \cdot \langle x', y'\rangle$$

$$= \vec{F}\cdot\vec{N}\,ds$$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \iint_D \operatorname{curl}\vec{F}\cdot\vec{k}\,dA$$

$$\int_C \vec{F}\cdot\vec{N}\,ds = \iint_D \operatorname{div}\vec{F}\,dA$$

equivalent forms in R^2

give rise to two distinct ~~forms~~ theorems in R^3

Stokes thm & Divergence theorem

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Ex $D = \{(x,y) : x^2+y^2 \le 1\}$

$P(x,y) = x \qquad Q(x,y) = xy$

$$\iint_D (Q_x - P_y)\,dx\,dy = \iint_D y\,dx\,dy = 0$$

by symmetry

$$\int_{\partial D} P\,dx + Q\,dy = \int_0^{2\pi} (\cos t \cdot -\sin t + \cos t \sin t \cdot \cos t)\,dt$$

$$= \left(\frac{\cos^2 t}{2} - \frac{\cos^3 t}{3}\right)\Big|_0^{2\pi} = 0.$$

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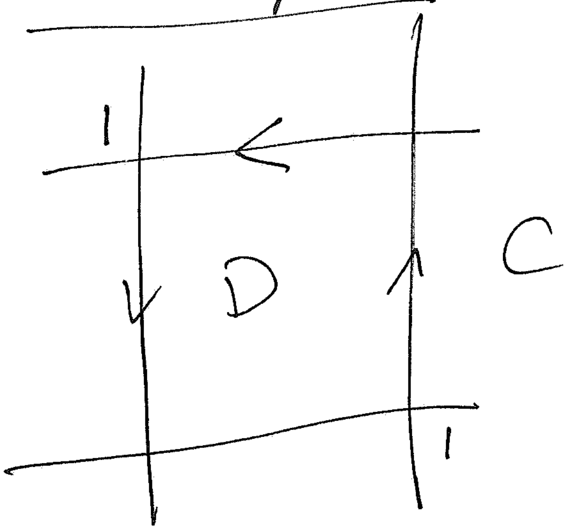
<u>Cor.</u> If $C = \partial D$ is simple

$$A(D) = \frac{1}{2} \int_{\partial D} x\,dy - y\,dx$$

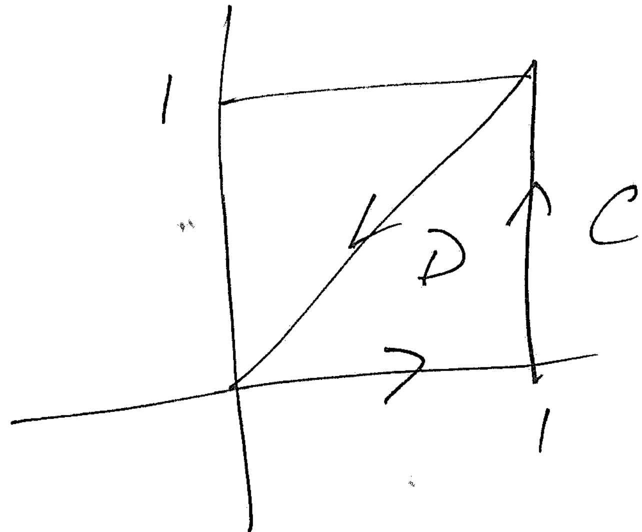
<u>Pf</u> $P = \frac{-y}{2}$ $Q = \frac{x}{2}$

$$Q_x - P_y = 1$$

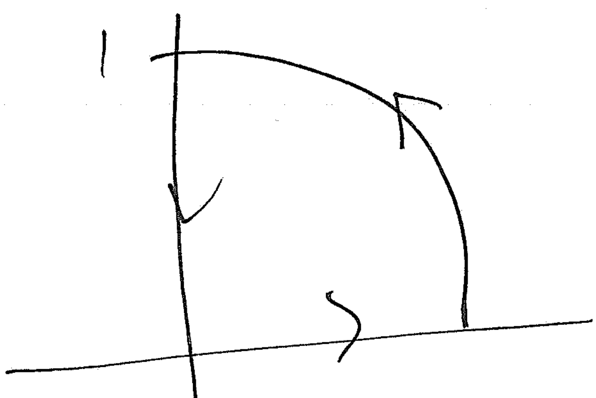
Examples



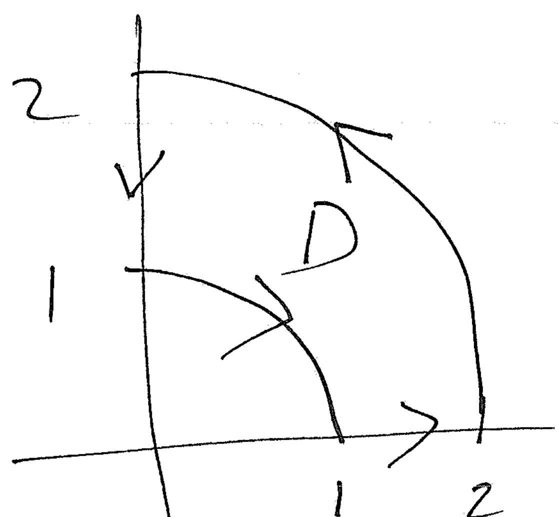
$$\int_C xy\,dx + (x^2 - y^2)\,dx$$



$$\int_C x^3\,dx - xy^2\,dy$$

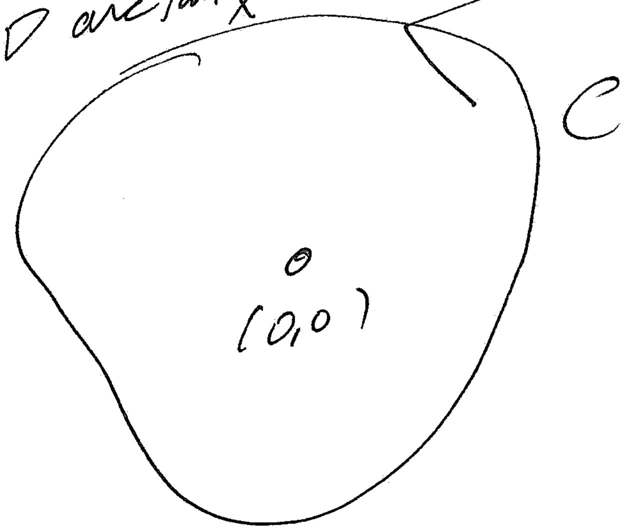


$$\int_C x\,dy - y\,dx$$

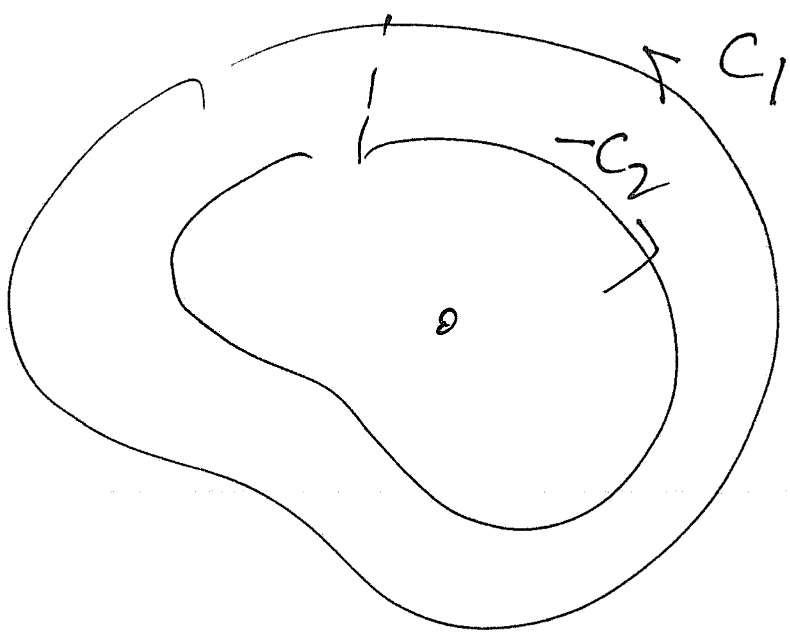


$$\int_C xy^2\,dx - x^2y\,dx$$

$$\oint_C \frac{x\,dy - y\,dx}{x^2+y^2} = \nabla \arctan \frac{y}{x}$$



C arbitrary



$$\int_{C_1} \frac{x\,dy - y\,dx}{x^2+y^2} = \int_{C_2} \frac{x\,dy - y\,dx}{x^2+y^2}$$

use Green's thm & fact

that $\vec{F} = \nabla \arctan y/x$

$$P = -\frac{y}{x^2+y^2} \qquad Q = \frac{x}{x^2+y^2} \qquad P_y = \frac{(x^2+y^2)(-1) + 2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$Q_x = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(\quad)^2}$$

Conservative vector fields in $\mathbb{R}^2$

Proposition If $\vec{F} = \nabla f$, $f \in C^1(D)$, then F is conservative in D.

Pf Let $P = \vec{c}(a)$, $Q = \vec{c}(b)$ be any path in D joining P to Q. Then

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

$$= \int_a^b \nabla f(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

$$\overset{\text{chain rule}}{=} \int_a^b \frac{d}{dt} f(\vec{c}(t))\, dt =$$

$$f(\vec{c}(b)) - f(\vec{c}(b))$$

So F is conservative. //

<u>Example 1</u> $\vec{F} = \frac{-y}{x^2+y^2}\,\vec{\imath} + \frac{x}{x^2+y^2}\,\vec{\jmath}$

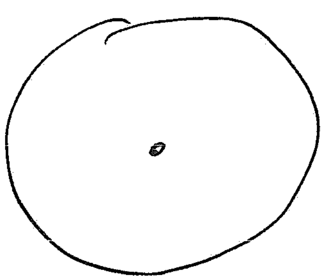
(note the singularity at $(0,0)$)

$= \nabla\theta \qquad \theta = \arctan\frac{y}{x}$

$\int_C \vec{F}\cdot d\vec{s} = 2\pi \qquad C =$ unit circle (counter clockwise)

$x = \cos t$
$y = \sin t$
$0 \le t \le 2\pi$

$$= \int_0^{2\pi} \left(-\sin t \cdot -\sin t + \cos t \cdot \cos t\right) dt = \int_0^{2\pi} 1\, dt = 2\pi$$



$\mathbb{R}^2 \setminus \{0,0\}$ not simple connected

Ex 2 $\vec{F} = c\dfrac{\vec{r}}{|r|^3}$ gravitational force field in $\mathbb{R}^3$

$\vec{r} = xi + yj + zk$

$\vec{F} = -c\,\nabla \dfrac{1}{r}$

Def'n A vector field $\vec{F}$ is conservative if $\int_{\vec{c}} \vec{F}\cdot d\vec{s}$ is indep't of the choice of path joining its end pts

The gravitational force field is conservative in $\mathbb{R}^3 \setminus \{(0,0,0)\}$!

Ex $\vec{F}(x,y,z) = \langle 3x^2z,\ z^2,\ x^3+2yz \rangle$

is conservative since

$\vec{F} = \nabla f$, $f = x^3 z + y z^2$

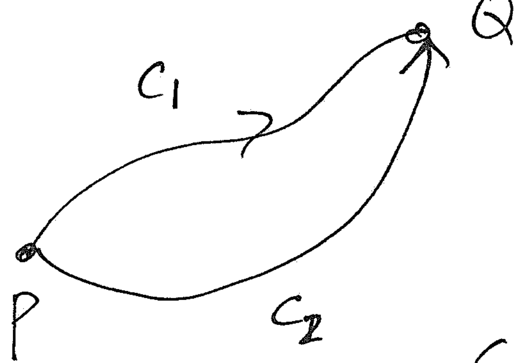
Proposition The following are equivalent (in a simple connected domain D)

1. $\vec{F}$ is conservative, i.e.
$\int_C \vec{F} \cdot ds$ is path indep't

2. $\int_C \vec{F} \cdot ds = 0$ $\forall$ closed paths C

3. $\vec{F} = \nabla f$

1) $\Rightarrow$ 2)



$$\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}$$

$$\Longleftrightarrow \int_{C_1 - C_2} \vec{F} \cdot ds = 0$$

2) $\Rightarrow$ 3)

Fix $P \in D$ and define

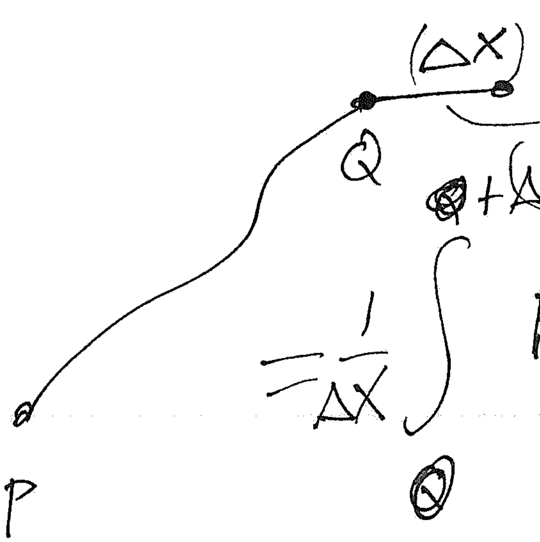
$$f(Q) = \int_C \vec{F} \cdot d\vec{s} \quad \text{where } C$$

is any path in D joining P to Q. This is well-defined!

$$f_x(Q) = \lim_{\Delta x \to 0} \frac{f(Q+\Delta x\vec{i}) - f(Q)}{\Delta x}$$

$$= \frac{1}{\Delta x}\left(\int_{C+\Delta x \vec{i}} \vec{F}\cdot d\vec{s} - \int_{C} \vec{F}\cdot d\vec{s} \right)$$

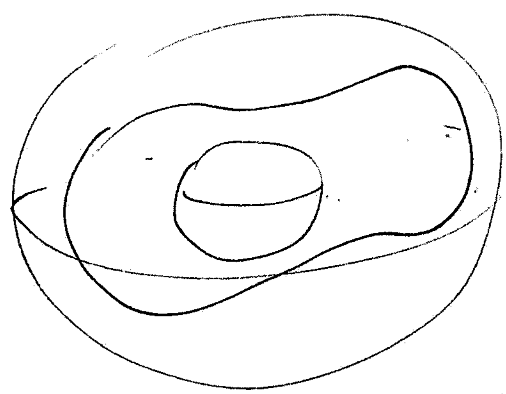
horizontal path $Q + t\vec{i}$, $0 \le t \le \Delta x$



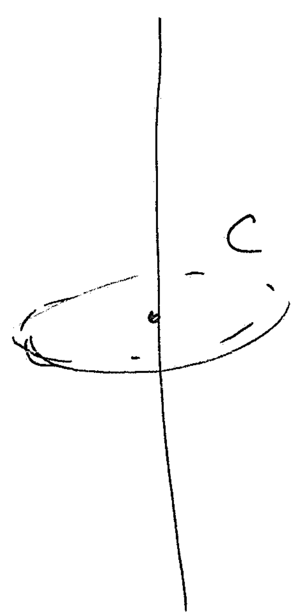
$$= \frac{1}{\Delta x}\int_{Q}^{Q+(\Delta x)\vec{i}} F_1(Q+t\vec{i})\, dt \xrightarrow{\Delta x \to 0} F_1(Q)$$

Note $\mathbb{R}^3 \setminus \{p_1 \cup \cdots \cup p_N\}$ is simply connected

$R_2 > R_1$ $\quad B_{R_2}(0) - B_{R_1}(0)$ is simply connected

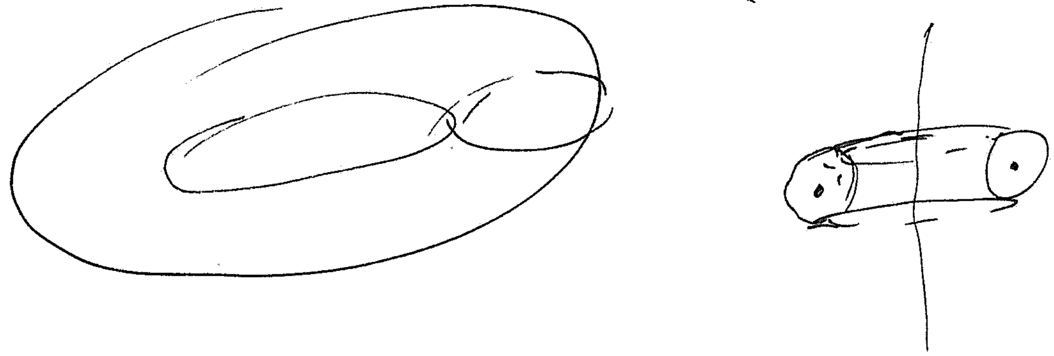


$\mathbb{R}^3 - \{z \text{ axis}\}$ is not simply connected



cannot contract C to a pt.

solid torus (inside ~~surface~~ a donut)



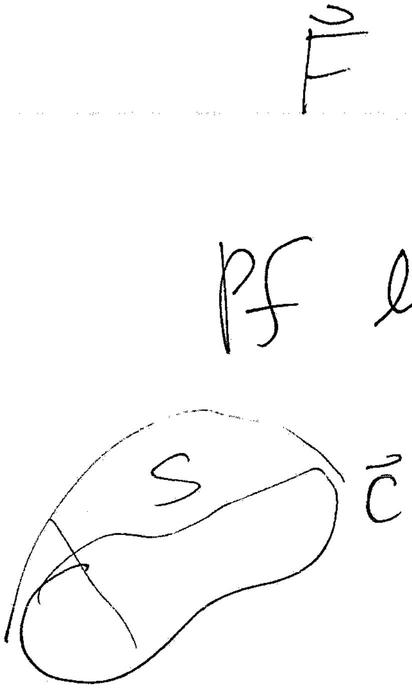
<u>Prop</u> curl $\nabla f = 0$

If D simply connected and

curl $\vec{F} = 0$ then

$\vec{F}$ is conservative

Pf later (Stokes theorem)



$$\oint_C \vec{F} \cdot ds = \int_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0$$