Math 202 Final exam study guide.

Know the definitions of dot product and its use in finding the projection of one vector onto another. Know how to find the cross product of two vectors (and the right hand rule) including its use in finding area and volume.

1. The directed vector going from $P = (p_1, p_2, p_3)$ to $Q = (p_1, p_2, p_3)$ is

$$\vec{PQ} =< q_1 - p_1, q_2 - p_2, q_3 = p_3 > \ .$$

2. Parametric equation of the line through $\vec{P} =< p_1, p_2, p_3 >$ in the direction of a vector $\vec{v} =< v_1, v_2, v_3 >$:

$$\vec{l}(t) =< x(t), y(t), z(t)) = \vec{P} + t\vec{v} =< p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 > \ .$$

Example: Find the parametric equation of the line through the points P=(1,-2,3) and Q=(-1,1,2).
Here we can take $\vec{v} = \vec{PQ} =< -2, 3, -1 >$ so $\vec{l}(t) =< 1, -2, 3 > +t < -2, 3, -1 >=< 1 - 2t, -2 + 3t, 3 - t >$. Note that when t=0 we are at P and when t=1 we are at Q. If we wanted to orient the line starting at Q and toward P, I would have taken $\vec{v} = \vec{QP}$. and started at Q.

3. The equation of the plane passing through $\vec{P}$ with normal direction $\vec{N}$ is given by

$$(\vec{X} - \vec{P}) \cdot \vec{N} = 0 \ .$$

Here $\vec{X} =< x, y, z >$ is the position vector of points on the plane and if $\vec{P} =< p_1, p_2, p_3 >$, $\vec{N} =< a, b, c >$, the equation of the plane is

$$a(x - p_1) + b(y - p_2) + c(z - p_3) = 0 \ .$$

Example: Find the equation of the plane passing through the points P=(0,0,0), Q=(1,1,1) and R=(0,-2,3). Then the vectors $\vec{PQ} = \vec{Q} =< 1, 1, 1 >$ and $\vec{PR} = \vec{R} =< 0, -2, 3 >$ lie in the plane (this means they are parallel to the plane or $\vec{Q} \cdot \vec{N} = \vec{R} \cdot \vec{N} =$

0. To find $\vec{N}$, we can use the cross product:

$$\vec{N} = \vec{Q} \times \vec{R} = < 5, -3, -2 > \ ,$$

so the equation of the plane is $5x - 3y - 2z = 0$.

Note also that the area of the triangle PQR is $\frac{1}{2}|\vec{N}| = \frac{\sqrt{38}}{2}$.

Suppose we are given a point such as T=(-1,2,4) that is not on the plane and we wanted to find the distance to this plane and the point U on the plane closest to T (this is the where the line through T in the direction of $\vec{N}$ intersects the plane). This line is $\vec{l}(t) = \vec{T} + t\vec{N} = < -1 + 5t, 2 - 3t, 4 - 2t >$; to find where it meets the plane we solve the linear equation

$$5(-1 + 5t) - 3(2 - 3t) - 2(4 - 2t) = 0 \text{ or } \quad -19 + 38t = 0,\ t = \frac{1}{2},\ U = (\frac{3}{2}, \frac{1}{2}, 3)\ .$$

Note that the distance from T to U is $|t\vec{N}| = |t||\vec{N}| = \frac{1}{2}\sqrt{38}$.
If we just wanted to find the distance fro T to the plane, we would use distance $= \frac{|\vec{T} \cdot \vec{N}|}{|\vec{N}|} = \frac{|-5-6-8|}{\sqrt{38}} = \frac{19}{\sqrt{38}} = \frac{\sqrt{38}}{2}$.

4. The directional derivative of a function $f(\vec{X}) = f(x, y, z)$ at $\vec{X_0}$ in the direction of a vector $\vec{v}$ is by definition $D_{\vec{v}} f(\vec{X_0}) = \lim_{h \to 0} \frac{f(\vec{X_0} + h\vec{v}) - f(\vec{X_0})}{h}$. Here we allow $\vec{v}$ to be of arbitrary length. However sometimes we will restrict $\vec{v}$ to be a unit vector, for example when we want to know the direction of fastest increase or decrease of f at $\vec{X_0}$. The directional derivative is easily seen to be equal to $\frac{d}{dt} f(\vec{l}(t))|_{t=0}$ where $\vec{l}(t)$ is the line through $\vec{X_0}$ in the direction of $\vec{v}$. When $\vec{v} = \hat{\imath},\ \hat{\jmath},\ \hat{k}$ respectively, then we obtain the partial derivative $f_x,\ f_y,\ f_z$ respectively. The gradient of f is the vector $\nabla f(x, y, z) = < f_x, f_y, f_z >$. The function f(x,y,z) is differentiable at (x_0, y_0, z_0) means that near $\vec{X_0}$, f(x,y,z) is well approximated by the linear approximation, that is ,

$$f(\vec{X_0} + \vec{h}) \approx f(\vec{X_0}) + \nabla f(\vec{X_0}) \cdot \vec{h}\ .$$

Example: Let $f(x, y) = x^2y^2 - x$ and let $(x_0, y_0) = (2, 1)$. Then $f_x = 2xy^2 - 1$, $f_y = 2x^2y$, $\nabla f(2, 1) = < 3, 8 >$. The tangent plane to the graph z=f(x,y) at (2,1, 2) is $z = 2 + 3(x - 2) + 8(y - 1)$ which represents the linear approximation to f(x,y) at (2,1). For example the linear approximation of $f(1.9, 1.1)$ is 2+3(-.1)+8(.1)=2.5.

A special case of the Chain rule tells us that

$$D_{\vec{v}}f(\vec{X_0}) = \nabla f(\vec{X_0}) \cdot \vec{v}\ .$$

More generally, if $\vec{c}(t)$ is a differentiable curve in $\mathbb{R}^3$, then

$$\frac{d}{dt}f(\vec{c}(t)) = \nabla f(\vec{c}(t)) \cdot \vec{c}'(t)\ .$$

In particular if $\vec{c}(0) = \vec{X_0}$, $\vec{c}'(0) = \vec{v}$, then

$$\frac{d}{dt}f(\vec{c}(t)) = \nabla f(\vec{X_0}) \cdot \vec{v} = D_{\vec{v}}f(\vec{X_0})\ .$$

Example: Let $f(x, y) = e^{1-x^2-2y^2}$. At the point (1,0), $\nabla f =< -2, 0 >$ and if $\vec{v} =< -1, 1 >$, $D_{\vec{v}}f(1, 0) =< -2, 0 > \cdot < -1, 1 >= 2$. The rate of change of f at (1,0) in the direction of the unit vector in the direction $< 1, -1 >$ is $\frac{2}{\sqrt{2}} = \sqrt{2}$. This represents the "slope" of f in that direction. The direction (unit) in which f increases the fastest is $\frac{\nabla f(1,0)}{|\nabla f(1,0)|} = \frac{<-2,0>}{2} =< -1, 0 >$. If $\vec{c}(t) =< 1 + t + \sin \pi t, 2t >$ then $\vec{c}(0) =< 1, 0)$, $\vec{c}'(0) =< 1 + \pi, 2 >$ and

$$\frac{d}{dt}f(\vec{c}(t))|_{t=0} = \nabla f(1, 0) \cdot \vec{c}'(0) =< -2, 0 > \cdot < 1 + \pi, 2 >= -2(1 + \pi)\ .$$

Now if $\vec{F}(x, y, z) =< F^1(x, y, z), F^2(x, y, z).F^3(x, y, z)$ is a vector function from $\mathbb{R}^3$ to $\mathbb{R}^3$, then its derivative DF at a point $\vec{X_0}$ is a linear map, i.e is represented as a 3x3 matrix called the Jacobian matrix:

$$\begin{pmatrix} F^1_x & F^1_y & F^1_z \\ F^2_x & F^2_y & F^2_z \\ F^3_x & F^3_y & F^3_z \end{pmatrix} (\vec{X_0})\ .$$

We can interpret the Jacobian matrix $D\vec{F}(\vec{X_0})$ as follows: given a vector $\vec{v}$, let $\vec{c}(t)$ be any curve in $\mathbb{R}^3$ passing through $\vec{X_0}$ with velocity vector $\vec{v}$, i.e $\vec{c}(0) = \vec{X_0}$, $\vec{c}'(0) = \vec{v}$. Then $\vec{F}(\vec{c}(t))$ is a new curve in $\mathbb{R}^3$ passing through $\vec{F}(\vec{X_0})$ with tangent vector $D_{\vec{v}}F(\vec{X_0} = D\vec{F}(\vec{X_0})(\vec{v})$ which we interpret as the matrix multiplication of the 3x3 Jacobian matrix and the 3x1 column vector $\vec{v}$.

A similar statement applies to any vector function $\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$ with m and n being 2 or 3.

Example: Let $\vec{F}(x,y,z) =< x^2-y+z, 2y-3z, x+z >$ and let $\hat{v} = \frac{1}{\sqrt{6}} < 2,-1,1 >$ be a unit vector. To calculate $D_{\hat{v}}\vec{F}(1,1,1)$, we first calculate the Jacobian matrix $D\vec{F}(1,1,1)$:

$$\begin{pmatrix} 2x & -1 & 1 \\ 0 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix} (1,1,1) = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix} .$$

Then $D_{\hat{v}}\vec{F}(1,1,1) = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 6 \\ -5 \\ 3 \end{pmatrix}$.

One can also compute $D_{\hat{v}}\vec{F}(1,1,1) =< D_{\hat{v}}F^1, D_{\hat{v}}F^2, D_{\hat{v}}F^3 > (1,1,1)$, that is component by component.

5. General Chain Rule: Let $\vec{G} : \mathbb{R}^n \to \mathbb{R}^p$ and $\vec{F} : \mathbb{R}^p \to \mathbb{R}^m$. Then the composition $\vec{h} = \vec{F} \circ \vec{G} : \mathbb{R}^n \to \mathbb{R}^m$ and

$$D\vec{h} = DF(\vec{G})D\vec{G} .$$

This last product is interpreted as matrix multiplication of a mxp matrix with a pxn matrix resulting in a mxn matrix.

Example: Let $f(x,y,z) = xyz$, $\vec{g}(u,v) =< u^2v, uv^2, u^2+v^2 >$ ($n=2$, $p=3$, $m=1$). Then $h(u,v) = f(\vec{g}(u,v))$ maps $\mathbb{R}^2 \to \mathbb{R}$. We compute ∇h as follows:

$$Df =< f_x, f_y, f_z >, \; D\vec{g} = \begin{pmatrix} g_u^1 & g_v^1 \\ g_u^2 & g_v^2 \\ g_u^3 & g_v^3 \end{pmatrix} .$$

Then,

$$Dh =< h_u, h_v >=< f_x, f_y, f_z > \begin{pmatrix} g_u^1 & g_v^1 \\ g_u^2 & g_v^2 \\ g_u^3 & g_v^3 \end{pmatrix} =< \nabla f \cdot \vec{g}_u, \nabla f \cdot \vec{g}_v > .$$

$$\nabla f =< yz, xz, xy >, \; \vec{g}_u =< 2uv, v^2, 2u >, \; \vec{g}_v =< u^2, 2uv, 2v > .$$

At $(u,v) = (1,2)$,

$$\vec{g}(1,2) =< 2,4,5 >, \; \nabla f \circ \vec{g} =< 20,10,8 >, \; \vec{g}_u =< 4,4,2 >, \; \vec{g}_v =< 1,4,4 > .$$

Therefore, $\nabla h =<< 20,10,8 > \cdot < 4,4,2 >, < 20,10,8 > \cdot < 1,4,4 >>=< 136, 92 >$.

6. We say a function of two or three variables is continuously differentiable of order k (we write $f \in C^k$) if f and all partial derivatives up to order k exist and are continuous. The important result is that for $k \geq 2$ the mixed derivatives are independent of the order. So for example if $f(x,y) \in C^2$ then $f_{xy} = f_{yx}$. Similarly if $f(x,y,z) \in C^2$, then $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$. Similarly if $f(x,y,z) \in C^4$, then $f_{xxyz} = f_{xyzx}$ etc.

This fact is used to derive the necessary condition $\nabla \times \vec{F}(x,y,z) = \vec{0}$ for the vector field $\vec{F}(x,y,z)$ to be conservative. For if $\vec{F} = \nabla\phi$, then

$$\nabla \times \vec{F} = \nabla \times \nabla\phi = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \phi_x & \phi_y & \phi_z \end{vmatrix} =< \phi_{zy} - \phi_{yz}, \phi_{xz} - \phi_{zx}, \phi_{yx} - \phi_{xy} >= \vec{0}\ .$$

Another important use of higher order partial derivatives is the Taylor expansion. For example if $f(x,y,z) \in C^2$ in a neighborhood of (x_0, y_0, z_0), then the Taylor expansion of f of order 2 at $\vec{X_0}$ is given by

$$f(\vec{X_0} + \vec{h}) = f(\vec{X_0}) + \nabla f(\vec{X_0}) \cdot \vec{h} + \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{X_0}) h_i h_j + E(\vec{X_0}, \vec{h})\ ,$$

where $\frac{E(\vec{X_0},\vec{h})}{|\vec{h}|^2} \to 0$ as $\vec{h} \to \vec{0}$. Here $\vec{h} =< h_1, h_2, h_3 >$ is the vector increment and must be small.

For a function $f(x,y)$ the Taylor expansion is considerably simpler:

$$\begin{aligned} f(x_0 + h_1, y_0 + h_2) &= f(x_0,y_0) + f_x(x_0,y_0)h_1 + f_y(x_0,y_0)h_2 \\ &+ \frac{1}{2}[f_{xx}(x_0,y_0)h_1^2 + 2f_{xy}(x_0,y_0)h_1h_2 + f_{yy}(x_0,y_0)h_2^2) + E\ . \end{aligned}$$

Example: Let $f(x,y)$ be a C^2 function with the property that (2,8) is a critical point of f, $f(-2,8) = -3$ and the Hessian matrix of f at (2,8) is

$$D^2 f(2,8) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}(2,8) = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}\ .$$

Then the second order Taylor polynomial of f at (2,8) is

$$-3 + \frac{1}{2}[-3h_1^2 + 2h_1h_2 - 2h_2^2]\ .$$

Sometimes we write $h_1 = x - x_0$, $h_2 = y - y_0$ so we get a second order polynomial in x,y as an approximation to $f(x, y)$ for (x,y) near (x_0, y_0):

$$f(x, y) \approx -3 + \frac{1}{2}[-3(x-2)^2 + 2(x-2)(y-8) - 2(y-8)^2] \ .$$

In this case at the critical point (2,8), the second derivative test says that (2,8) is a relative maximum.

7. Implicit surfaces $g(x, y, z) = 0$ and the Implicit function theorem.

Let $g(x, y, z)$ be a differentiable function of three variables; the the level set

$$S = \{(x, y, z) : g(x, y, z) = 0\}$$

implicitly defines a surface S in $\mathbb{R}^3$, possibly with singularities. The implicit function theorem states that if $\nabla g(\vec{X_0}) \neq \vec{0}$, then S is a regular surface in some neighborhood of $\vec{X_0}$. In particular if $g_z(\vec{X_0}) \neq \vec{0}$ then S is locally a graph, i.e the implicit equation $g(x, y, z) = 0$ locally has a unique solution $z = z(x, y)$ with $z(x_0.y_0) = z_0$. Therefore $z(x, y)$ satisfies $g(x, y, z(x, y)) = 0$ for (x,y) near $(x_0.y_0)$. We can differentiate this identity with respect to x and y:

$$g_x + g_z z_x = 0, \ g_y + g_z z_y = 0 \ .$$

This leads to

$$z_x = -\frac{g_x(x, y, z(x, y))}{g_z(x, y, z(x, y))}, \ z_y = -\frac{g_y(x, y, z(x, y))}{g_z(x, y, z(x, y))} .$$

Example: Show that $g = xy + z + 3xz^5 - 4 = 0$ is solvable for z as a function of (x,y) near (1,0,1) and compute z_x, z_y at (1,0). We compute $g_z(1, 0, 1) = 1 + 15 = 16 > 0$ so the IFT says that we can solve for $z = z(x, y)$ near the given point. By the above formulas,

$$z_x = -\frac{3}{16}, \ z_y = -\frac{1}{16} \text{ at } (1,0) \ .$$

Now let $\vec{c}(t)$ be a differentiable curve that lies on S, that $g(\vec{c}(t)) = 0$ with $\vec{c}(0) = \vec{X_0}$ and $\vec{c}'(0) = \vec{v}$. Assume $\nabla g(\vec{X_)}) \neq \vec{0}$. Then by the Chain Rule,

$$0 = \frac{d}{dt} g(\vec{c}(t))|_{t=0} = \nabla g(\vec{X_0}) \cdot \vec{v} \ .$$

Since $\vec{c}(t)$ lies on S and S is regular near $\vec{X_0}$, the vector $\vec{v}$ lies in the tangent plane to S at $\vec{X_0}$. (The tangent plane to a regular surface S at $\vec{X_0}$ is the set of tangent vectors to regular curves on S at $\vec{X_0}$.) Since $\nabla g(\vec{X_0}) \cdot \vec{v} = 0$ for all such tangent vectors $\vec{v}$, we conclude that $\nabla g(\vec{X_0})$ is the nornal vector to the surface S at $\vec{X_0}$.

Example: Find the outward unit normal to the surface $x^2+y^2+2z^2 = 4$ at (1,1,1). Here $g(x,y,z) = x^2 + y^2 + 2z^2 - 4$ and $\nabla g(1,1,1) =< 2,2,4 >$. There the outward unit normal is $\hat{N} = \frac{1}{\sqrt{24}} < 2,2,4 >$.

8. The method of Lagrange multipliers.

Consider the problem of finding the extremum of a function $f(x,y,z)$ subject to a constraint $g(x,y,z) = 0$. A point $\vec{X_0}$ which is a possible extremum (max or min) no longer need satisfy the condition $\nabla f(\vec{X_0}) = 0$. Instead, the method of Lagrange multipliers says that if $\nabla g(\vec{X_0}) \neq 0$, there is a number λ such that

$$\nabla f(\vec{X_0}) = \lambda \nabla g(\vec{X_0}), \text{ and } g(\vec{X_0})) = 0 \ .$$

To see this we let $\vec{c}(t)$ be a differentiable curve such that $g(\vec{c}(t)) = 0$ with $\vec{c}(0) = \vec{X_0}$. Then $f(\vec{c}(t)$ has a min or max at $t = 0$ so

$$\frac{d}{dt} f(\vec{c}(t)|_{t=0} = 0, \text{ or } \nabla f(\vec{X_0}) \cdot \vec{c}\,'(0) = 0 \ .$$

In other words, $\nabla f(\vec{X_0})$ is normal to the level set $g(x,y,z) = 0$ at $\vec{X_0}$. But since $\nabla g(\vec{X_0})$ is the nornal vector at $\vec{X_0}$ we must have $\nabla f(\vec{X_0}) = \lambda \nabla g(\vec{X_0})$ (these vectors are parallel).

Example: Find the absolute max and min of $f(x,y,z) = x + y - z$ on the ball $x^2 + y^2 + z^2 \leq 1$.
Since $\nabla f =< 1,1,-1 > \neq \vec{0}$, there are critical points inside the ball . We use Lagrange multipliers on the boundary of the ball. That is we look for the max and min of f subject to the constraint $g(x,y,z) = x^2 + y62 + z^2 - 1 = 0$. Therefore,

$$\nabla f =< 1,1,-1 >= \lambda \nabla g = \lambda < 2x, 2y, 2z > \ .$$

This implies $(2\lambda x)^2 + (2\lambda y)^2 + (2\lambda z)^2 = 1$. Since $x^2 + y^2 + z^2 = 1$, this gives $4\lambda^2 = 3$ so $\lambda = \pm\frac{\sqrt{3}}{2}$ Therefore we have two extremum points:

$$P =< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} >, \ f(P) = \sqrt{3}, \text{ absolute max}$$

$$Q =< -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >,\ f(Q) = -\sqrt{3}, \text{ absolute min }.$$