Kepler's Laws

The German astronomer Kepler discovered three fundamental laws governing planetary motion. Kepler's first law is that planetary motion is ellipitcal with the sun at one focus (the motion is planar). His second law is that equal areas of the position vector from the sun to the planet are swept out in equal times. Kepler's third law is that the period T of the motion satisfies $T^2 = Ka^3$ for a universal constant K where a is the major semi axis of the ellipse. (Surprisingly, T is independent of the minor semi axis b of the ellipse). Kepler's laws were based on the careful data of his mentor Tycho Brahe and represent a profound discovery. However the explanation for these laws provided by Newtonian mechanics is one of the great achievements of science.

Let's start with the second law which is easiest to explain using vector calculus. The change in position is $\Delta\vec{r} = \vec{r}'(t)\Delta t$ so the change in area swept out in time Δt is

$$\Delta A = \frac{1}{2}|\vec{r}(t) \times \vec{r}'(t)|\Delta t$$

since to first order this region is a triangle with sides $\vec{r}(t)$, $\vec{r}(t+\Delta t)$, Δr. Note however that the motion lies in a plane which means that both $\vec{r}(t)$ and $\vec{r}'(t)$ lie in the plane of motion. Hence $\vec{N}(t) := \vec{r}(t) \times \vec{r}'(t)$ is normal to the plane of motion. Thus Kepler's second law is equivalent to showing that N is constant.

To this end, consider

$$\text{(1)} \qquad \frac{d\vec{N}}{dt} = \vec{r}' \times \vec{r}' + \vec{r} \times \vec{r}'' = \vec{r} \times \vec{r}'' = \vec{r} \times \vec{a}\,,$$

where $\vec{a} = \vec{r}''$ is the acceleration vector of the motion. Newton's famous law $\vec{F} = m\vec{a}$ (here $\vec{F}$ is the force of the gravitational attraction of the sun and the planet). So (1) says that $\frac{d}{dt}(\vec{r} \times \vec{r}') = 0$ if and only if $\vec{r} \times \vec{F} = 0$, that is $\vec{F}$ is parallel to $\vec{r}$. This is what's known as a central force. Newton's law of universal gravitation (inverse square law) is more precise:

$$\text{(2)} \qquad \vec{F} = -MmG\frac{\vec{r}}{|\vec{r}|^3}$$

where M is the mass of the sun, m is the mass of the planet and G is the universal gravitational constant. So Kepler's second law follows from (2).

The proof of Kepler's first law using (2) is somewhat tricky. Write $\vec{r} = r\hat{r}$ where $|\hat{r}| = 1$. Then $\vec{r}' = r'\hat{r} + r\hat{r}'$. Since $\hat{r}^2 = 1$, $\hat{r} \cdot \hat{r}' = 0$. Here we can interpret $\hat{r}' = \omega\hat{T}$ ($|\hat{T}| = 1$) as the angular velocity vector (ω is the scalar angular velocity). Continuing,

$$\vec{a} = \vec{r}'' = r''\hat{r} + 2r'\hat{r}' + r\hat{r}'' = -GM\frac{\hat{r}}{r^2} \tag{3}$$

by (2). Taking the dot product of both side of (3) with $\hat{r}$ gives

$$r'' - r\omega^2 = -\frac{GM}{r^2} \ . \tag{4}$$

since $\hat{r} \cdot \hat{r}'$ implies $\hat{r} \cdot \hat{r}'' = -\omega^2$.

Recalling $|\vec{r} \times \vec{r}'| = |r\hat{r} \times (r\hat{r}' + r'\hat{r})| = |r^2\hat{r} \times \hat{r}'| = r^2\omega$ is constant, we can write that the angular momentum $mr^2\omega = L$ constant. Inserting this in (4) gives

$$r'' = -\frac{GM}{r^2} + \frac{L^2}{m^2r^3} \ . \tag{5}$$

So far so good. Now comes the tricky part. Motivated by "knowing that" the orbit is an ellipse with one focus at the sun, it makes sense to introduce polar coordinates (r, θ) with center at the sun. What does the equation of an ellipse look like in these coordinates? With respect to standard x,y coordinates , let the ellipse have major semi axis a and minor semi axis b with the sun at (c,0). Then $e = \frac{c}{a}$ is the eccentricity of the ellipse and the equation of the ellipse becomes

$$r + |\vec{r} + 2c\hat{i}| = 2a \ . \tag{6}$$

Therefore,

$$(r - 2a)^2 = r^2 + 4cr\cos\theta + 4c^2 \ .$$

Simplifying leads to

$$\frac{1}{r} = \frac{a + c\cos\theta}{a^2 - c^2} = \frac{1 + e\cos\theta}{a(1 - e^2)} \ . \tag{7}$$

Now introduce $u = \frac{1}{r}$ and θ as new independent variables. Then

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \omega\frac{d}{d\theta} = \frac{Lu^2}{m}\frac{d}{d\theta} \ , \tag{8}$$

$$\frac{dr}{dt} = \frac{1}{u^2}\frac{du}{dt} = -\frac{L}{m}\frac{du}{d\theta} \ , \tag{9}$$

$$\frac{d^2r}{dt^2} = -\frac{L^2}{m^2}u^2c\frac{d^2u}{d\theta^2} \ . \tag{10}$$

Inserting (8)(9)(10) into (5) and simplifying yields

$$u_{\theta\theta} + u = \frac{GM}{L^2}m^2 \ . \tag{11}$$

But (11) is easy to solve. The general solution is

$$u = \frac{GM}{L^2}m^2 = A\cos\theta + B\sin\theta \ .$$

To find A,B we must understand the initial conditions for u, $\frac{du}{d\theta}$ at $\theta = 0$. In terms of the motion at $\theta = 0$ we are at the perihelion or closest position to the sun, hence r achieves its minimum value or $u = \frac{1}{r}$ achieves its maximum value. Hence $B = \frac{du}{d\theta}(0) = 0$ and so

$$\frac{1}{r} = \frac{GM}{L^2}m^2 = A\cos\theta \ .$$

To compare this with the form of the ellipse we derived earlier in (7), we rewrite this as

$$\frac{1}{r} = \frac{1 + A'\cos\theta}{\frac{L^2}{GMm^2}}$$

for a new constant A' satisfying

$$r_{\text{min}}(1 + A') = \frac{L^2}{GMm^2} \ .$$

Then $A' = e$, $\frac{L^2}{GMm^2} = a(1-e^2)$ and a,e are determined by the equations

$$a(1-e) = r_{\text{min}} \tag{12}$$

$$a(1-e^2) = \frac{L^2}{GMm^2} \tag{13}$$

This concludes the proof of Kepler's first law.

We now turn to Kepler's third law. The area of the elliptical orbit with semi major axis a and semi minor axis b $((\frac{b}{a}0^2 = 1 - e^2)$ is πab by a standard Calculus 2 computation. On the other hand,

$$\frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m} \ .$$

Hence if T is the period of the orbit, $\frac{L}{2m}T = \pi ab$. This gives

$$T^2 = (\frac{2m\pi ab}{L})^2 = \frac{4m^2\pi^2a^2b^2}{L^2} = \frac{4m^2\pi^2a^4(1-e^2)}{GMm^2a(1-e^2)} \ ,$$

or $T^2 = \frac{4\pi^2a^3}{GM}$. Note that T is independent of b!