When does a vector field have a potential function?

Consider a vector field (for example a force field) $\vec{F}(\vec{x}) =< F^1(\vec{x}), F^2(\vec{x}), F^3(\vec{x}) >$ where $\vec{x} =< x_1, x_2, x_3 >$ and F^1, F^2, F^3 are the coordinate functions of $\vec{F}$. We say that $\vec{F}$ has a potential function ϕ or is **conservative** if $\vec{F} = \nabla\phi$. If $\vec{F}$ is C^1 and ϕ is C^2 , then by the theorem of the equality of mixed partial derivatives applied to ϕ, a necessary condition is that

$$\frac{\partial F^i}{\partial x_j} = \frac{\partial F^j}{\partial x_i} \text{ for all } i \neq j \ .$$

There is an important example of a vector field in R^2 where this "local condition" is satisfied (which actually implies that in a neighborhood of any point there exists a potential function) but fails to have a "global potential" function.

Example 0.1. Let $\vec{V}(x,y) =< -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} >$. Then it is easy to check that

$$\frac{\partial}{\partial y}(-\frac{y}{x^2+y^2}) = \frac{\partial}{\partial x}(\frac{x}{x^2+y^2}) \ ,$$

so the necessary condition is satisfied at every point except (0,0) where $V(x,y)$ is not defined. To understand what is going on, we can write $\vec{V} = \nabla\phi$ in polar coordinates:

$$\vec{V}(r,\theta) = \frac{1}{r} < -\sin\theta, \cos\theta >= \frac{1}{r}\hat{e_\theta} = \phi_r \hat{e_r} + \frac{\phi_\theta}{r}\hat{e_\theta} \ .$$

(Here we are using our previous example of how $\nabla\phi$ looks in polar coordinates.) Comparing these expressions tells us that $\phi_r = 0$ and $\phi_\theta = 1$. Thus up to a constant, $\phi = \theta = \arctan\frac{y}{x}$, i.e ϕ is just the angle θ. But this is a problem if we want ϕ to be defined in a neighborhood of the unit circle, for such an angle must be discontinuous when we start at the point (0,1) and move counterclockwise around the circle and return to our starting point. However there is no problem locally in defining such an angle. We will return to this problem later in the semester.