1. Suppose that $u(x,t)$ satisfies the inhomogeneous heat equation
\[ u_t = u_{xx} - 2 \quad \text{for } 0 < x < 2, \ t > 0 \]
with boundary conditions $u(0, t) = 4$, $u(2, t) = 0$ and initial condition $u(x, 0) = 4 + x^2$.

a. (10pts) Find the steady state temperature (say $v(x)$).

\[ v_{xx} = 2, \ v(0) = 4, \ v(2) = 0 \implies v(x) = (x - 2)^2. \]

b. (15pts) Find $u(x, t)$.

Let $\tilde{u} = u(x, t) - v(x)$. Then
\[ \tilde{u}_t = \tilde{u}_{xx} = 0, \ \tilde{u}(0, t) = \tilde{u}(2, t) = 0, \ \tilde{u}(x, 0) = 4 + x^2 - (x - 2)^2 = 4x. \]

\[ \tilde{u}(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2} e^{-\left(\frac{n\pi}{2}\right)^2 t}, \]

where $B_n = \int_0^2 4x \sin \frac{n\pi x}{2} dx = \frac{16}{n\pi} (-1)^{n+1}$. Therefore,
\[ u(x, t) = (x - 2)^2 + \sum_{n=1}^{\infty} \frac{16}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{2} e^{-\left(\frac{n\pi}{2}\right)^2 t}. \]

2. Let $f(x) = (x - 1)^2$ for $0 \leq x \leq 1$.

a. (11pts) Compute the Fourier cosine series of $f(x)$.

\[ f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x \]

where
\[ a_0 = \int_0^1 (x - 1)^2 \, dx = \frac{1}{3} \quad \text{and} \]
\[ a_n = 2 \int_0^1 (x - 1)^2 \cos n\pi x \, dx = \frac{4}{(n\pi)^2}. \]

b. (7pts) Draw a careful graph of the function to which your series converges to on the interval $-2 \leq x \leq 2$. (Note that I am not asking you to graph the Fourier cosine series!)

Extend $f(x)$ to (-1,0) as an even function $\tilde{f}$, i.e. $\tilde{f}(x) = f(-x)$ for $x \in (-1,0)$ and then extend $\tilde{f}$ everywhere as a periodic function of period 2. The fourier cosine series converges to this extension.

c. (7pts) Do not compute the Fourier sine series of $f(x)$ but draw a careful graph of the function to which the Fourier sine series converges on the interval $-2 \leq x \leq 2$. (Note that I am not asking you to graph the Fourier sine series!)

Similarly extend $f(x)$ to (-1,0) as an odd function $g(x)$, i.e. $g(x) = -g(-x)$ for $x \in (-1,0)$ and then extend $g$ everywhere as a periodic
function of period 2. The Fourier sine series converges to this extension.

3. (20 pts) Solve $\Delta u = 0$ in the unit disk $B_1(0) \subset \mathbb{R}^2$ (using Fourier series in polar coordinates) with boundary condition $u(1, \theta) = |\theta|$, $-\pi \leq \theta \leq \pi$.

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) .$$

When $r = 1$, $u = |\theta|$ on $(-\pi, \pi)$ (and extended periodic) which is even so $B_n = 0$ and

$$A_0 = \frac{1}{\pi} \int_{0}^{\pi} \theta \, d\theta = \frac{\pi}{2} ;$$

$$A_n = \frac{2}{\pi} \int_{0}^{\pi} \theta \cos n\theta \, d\theta$$

$$= \frac{2}{\pi} \left\{ \theta \frac{\sin(n\theta)}{n} \bigg|_{0}^{\pi} - \int_{0}^{\pi} \frac{\sin(n\theta)}{n} \, d\theta \right\}$$

$$= \frac{2 \cos n\theta}{\pi \, n^2} \bigg|_{0}^{\pi} = \frac{2}{n^2 \pi} \begin{cases} 0 & \text{if } n \text{ even} \\ -2 & \text{if } n \text{ odd} \end{cases}$$

Hence

$$u(r, \theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} r^n \cos \frac{n\theta}{n^2} .$$

4. (30 pts) Solve using the D’Alembert formula and Duhamel’s principle:

$$u_{tt} = u_{xx} + xt \quad \text{on} \quad -\infty < x < \infty , \ t > 0 , \ u(x, 0) = e^x , \ u_t(x, 0) = 0 .$$

By superposition and D’Alembert, $u(x, t) = \frac{1}{2} (e^{x+t} + e^{x-t}) + v(x, t)$ where $v$ satisfies

$$v_{tt} = v_{xx} + xt , \ v(x, 0) = v_t(x, 0) = 0 .$$

By Duhamel’s principle, $v(x, t) = \int_{0}^{t} w(x, t; s) \, ds$, where

$$w_{tt} = w_{xx} , \ w(x, 0) = 0 , \ w_t(x, s) = xs .$$

Hence

$$w(x, t; s) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} sy \, dy = \frac{s}{4} [(x+(t-s))^2 - (x-(t-s))^2] = sx(t-s) .$$

This gives

$$v(x, t) = \int_{0}^{t} xs(t-s) \, ds = x \int_{0}^{t} (ts - s^2) \, ds = x \left( \frac{t^3}{2} - \frac{t^3}{3} \right) = \frac{x \, t^3}{6} .$$